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The Knightian Uncertainty Hypothesis: Unforeseeable Change and Muth's Consistency Constraint in Modeling Aggregate Outcomes

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Abstract

This paper introduces the Knightian Uncertainty Hypothesis (KUH), a new approach to macroeconomics and finance theory. KUH rests on a novel mathematical framework that characterizes both measurable and Knightian uncertainty about economic outcomes. Relying on this framework and John Muth's pathbreaking hypothesis, KUH represents participants'

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forecasts to be consistent with both uncertainties. KUH thus enables models of aggregate outcomes that 1) are premised on market participants' rationality, and 2) yet accord a role to both fundamental and psychological (and other non-fundamental) factors in driving outcomes. The paper also suggests how a KUH model's quantitative predictions can be confronted with time-series data.

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1 Introduction and Overview

In his classic book *Risk, Uncertainty, and Profit*, Frank Knight introduced a distinction between measurable uncertainty, which he called "risk," and "true uncertainty," which cannot "by any method be reduced to an objective, quantitatively determined probability" (Knight, 1921, p. 321). Knight argued that "true uncertainty" arises from change that cannot be fully foreseen with probabilistic rules and whose consequences for market outcomes, and thus payoffs from market participants' decisions, cannot be fully comprehended – even in hindsight. For Knight, recognizing such unforeseeable change is the key to understanding profit-seeking activity in real-world markets.

The rational expectations hypothesis (REH) and behavioral finance are widely considered to have been the milestones in the development of models of aggregate outcomes, resulting from market participants' decisions, since the 1970s.[1] Although they differ in essential respects, the REH and behavioral-finance approaches share a key feature: their models specify aggregate outcomes with a stochastic process.[2] By design, these models assume that economists do not face Knightian uncertainty. By contrast, the Knightian uncertainty hypothesis (KUH) proposed here enables economists to build models that acknowledge their own Knightian uncertainty stemming from unforeseeable change in the process driving outcomes.

Recognizing uncertainty that cannot be represented with standard probabilistic measures of risk is increasingly viewed as crucial to remedying shortcomings of macroeconomic and finance theory. For example, in his Nobel lecture, Hansen (2013, p. 399, emphasis added) argues that REH models "miss something *essential*: uncertainty [arising from] ambiguity about which is the correct model" of the process driving aggregate outcomes.

Following a pioneering contribution by Hansen and Sargent (2008), a number of recent papers build macroeconomic models that recognize ambiguity on *the part*

[1] Lucas's (1972a,b) early contributions are usually cited as pioneering the application of REH to macroeconomic theory. For authoritative surveys of the behavioral-finance approach, see Shleifer (2000) and Barberis and Thaler (2003).

[2] Akerlof and Shiller (2009) is a notable exception in behavioral-finance literature. They rely on a narrative mode of analysis, and thus *ipso facto* avoid formalizing behavioral findings with models specifying aggregate outcomes with a stochastic process.

of market participants. Although such models relate aggregate outcomes to participants' demand and supply decisions, they represent these outcomes with a stochastic process. They typically do not relate ambiguity to unforeseeable change in the processes driving outcomes.

In a significant departure from this literature, Ilut and Schneider (2014) open the New Keynesian (NK) model to unforeseeable change in the process driving the model's exogenous variable – total factor productivity (TFP). However, Ilut and Schneider constrain precisely – with a probabilistic rule – their model's representation of participants' forecasts of TFP and how these forecasts drive aggregate outcomes (for example, hours worked and the inflation rate). As a result, they represent how aggregate outcomes unfold over time with a stochastic process, thereby assuming that, unlike participants, economists do not face ambiguity about the process driving these outcomes.

In this paper, we propose a new approach to building models of aggregate outcomes that removes this incongruity from macroeconomic and finance theory. Our approach, which we call the Knightian Uncertainty Hypothesis (KUH), recognizes that, like market participants, economists also face Knightian uncertainty and the ambiguity that such uncertainty engenders.

Like REH and behavioral finance, a KUH model represents the process driving outcomes at a point in time with a stochastic process. However, in contrast to these approaches, KUH rests on a novel mathematical framework that formalizes *both* measurable and Knightian uncertainty about the process driving aggregate outcomes. By assuming that, over time, a macroeconomic model's coefficients undergo unforeseeable change, KUH opens economists' models to Knightian uncertainty.[3]

By design, therefore, a KUH model does not represent outcomes with a stochastic process, which rules out reliance on the standard (conditional) expectation

[3] An overwhelming majority of REH and behavioral-finance models constrain their specifications – the set of variables that they include and their coefficients – to be unchanging over time. Whenever these models recognize that the process underpinning outcomes undergoes change, they represent such change with a probabilistic rule, such as Markov switching, determining completely how the model's specifications unfold over time. For a seminal development of models representing change with probabilitic rules and an authoritative recent review, see Hamilton (1988, 2008). .

to define the model's predictions of future outcomes. Instead, KUH's characterization of Knightian uncertainty enables us to define a hitherto unavailable conception of prediction applicable to outcomes that undergo unforeseeable change. To ensure consistency between its models' characterization of both kinds of uncertainties and their representation of participants' forecasts, KUH relies on Muth's (1961) path-breaking hypothesis.

Muth (1961) argued that market participants are "rational" in the dictionary sense: they have some understanding – *albeit imperfect* – of the process driving outcomes, and they make use of this understanding in pursuing their objectives, typically profit-seeking.[4] Importantly, Muth advanced the seminal hypothesis that an economist can represent participants' diverse understandings of the process driving aggregate outcomes with his own mathematical model of this process. Consequently, Muth proposed that an economist specify participants' forecasts of outcomes by constraining them to be consistent with his model's predictions of these outcomes.

Muth emphasized that his hypothesis "*does not* assert that the scratch work of entrepreneurs resembles [an economic model's] system of equations in any way." (1961, p. 317, emphasis in the original). However, he believed that, although boldly abstract, his hypothesis offered a "sensible" way to *acknowledge* participants' rationality – that their forecasts are related to their understanding of "the way the economy works" – in economic models (1961, p. 315).[5] After all, the very meaning of model building is that it formalizes an economist's hypothesis about the process driving aggregate outcomes and how they *actually* unfold over time.

Invoking this hypothesis, Lucas argued in the early 1970s that representing market participants' forecasts of aggregate outcomes to be inconsistent with the model's

[4] According to the Merriam-Webster Dictionary, an individual is "rational [if he] has a latent or active power to make logical inferences and draw conclusions that enable [him] to understand the world about him and relate such knowledge to the attainment of ends."

[5] Muth explicitly contrasted his hypothesis with Simon's (1959) "bounded rationality" approach. Bounded rationality assumes that, faced with insuperable obstacles to understanding the structure of the economy, participants rely on forecasting rules – for example, adaptive expectations – that are not explicitly related to "the way the economy works." Muth (1961, p.316) stressed that his "hypothesis is based on exactly the opposite point of view: that dynamic economic models do not assume enough rationality" on the part of market participants.

predictions of them presumes *ex ante* that participants will ignore forecast errors, thereby foregoing profit opportunities time and again over an indefinite future. As Lucas recounts in his Nobel lecture (1995, p. 255), the implication that inconsistent models presume participants' irrationality played a crucial role in persuading macroeconomists to embrace REH.[6]

REH implements Muth's hypothesis in models representing outcomes with a stochastic process. Although doing so eliminates irrationality, it also *fully* constrains an REH model's representation of participants' forecasts. Thus, once an economist assumes that market participants are rational and that he does not face Knightian uncertainty, he can no longer accord participants' forecasts an autonomous role in driving aggregate outcomes. As Sargent put it in his interview with Evans and Honkapohja (2005, p. 566): "in rational expectations models, people's beliefs are among the outcomes of our [economists'] theorizing. They are not inputs" to an economist's model.

As in REH models, applying Muth's hypothesis in a KUH model constrains the model's representation of participants' forecasts in terms of the model's coefficients and moments of its stochastic innovations. However, because a KUH model recognizes that an economist faces Knightian uncertainty, imposing consistency within the model does *not* fully constrain its representations of forecasts. The KUH approach thereby uncovers the key implication of Knightian uncertainty for building macroeconomic models: an economist faces ambiguity about how rational participants forecast outcomes and make decisions.[7]

[6]Lucas (1995, p. 255 and 2001, p.13) emphasized that macroeconomic models of the 1960s, which relied on adaptive expectations, were based on a "glaring" inconsistency, and thus were "the wrong theory" of time-series regularities. For an extensive discussion of this revolutionary development in macroeconomic theory, see Frydman and Phelps (2013). For a formal illustration of Lucas' point and further discussion of how his arguments apply to behavioral-finance models, see Sections 5 and 8.3.

[7]Analogously to its role in REH models, Muth's hypothesis enables economists to make use of calibration in confronting a KUH model with time-series data. However, because a KUH model does not fully constrain the relationship between parameters characterizing participants' preferences and/or technology and model-implied coefficients of aggregate outcomes, microeconomic estimates of such parameters cannot be used to calibrate the model. This implication of Knightian uncertainty underscores the importance of Hansen and Heckman's (1996, p. 90) argument that the calibration methodology should be based on an "explicit econometric framework." We provide an illustration of how this can be done in a KUH model in Section 9 and Appendix B.

By recognizing this ambiguity, KUH opens a way to build macroeconomic models that accord participants' *diverse* forecasts an autonomous role in driving aggregate outcomes, without presuming that participants are irrational.[8] In this sense, the KUH approach enables economists to realize the vision that motivated Phelps's (1970, p. 22) pioneering micro-foundations approach: because market participants "maximize relative to their" own imperfect and diverse understandings of how the economy works, their forecasts play an autonomous role in driving aggregate outcomes, such as the inflation and unemployment rates. By ruling out such a role for participants' forecasts, the REH approach preempted this vision.[9]

Moreover, reliance on models that represent outcomes with a stochastic process has rendered the REH and behavioral finance approaches irreconcilable on *logical grounds*. The *raison d'être* of behavioral finance is that psychological and other non-fundamental factors exert an autonomous influence on participants' forecasts. However, in order to accord these forecasts an autonomous role in a model specifying aggregate outcomes with a stochastic process, an economist must rely on inconsistent representations of participants' forecasts.

Thus, once an economist hypothesizes that a stochastic process can represent how outcomes unfold over time, he can follow either the REH or the behavioral-finance approach. However, he cannot build models that synthesize the core ideas underpinning each of these approaches.[10]

KUH reveals a novel way to build models that rest on a synthesis of Muth's hy-

[8] Diversity refers to differences in how market participants interpret the relationship between time-t information and future outcomes, though the KUH approach also allows for heterogeneity of information about fundamental factors to which market participants have access. In this sense, the diversity in our KUH prototype model arises from recognizing that an economist faces Knightain uncertainty. We assume here that participants have access to the same information about fundamentals, such as corporate earnings or productivity, and leave for future research the development of a KUH intertemporal model that allows for both diversity and informational asymmetry.

[9] For an early discussion of how REH supplanted the core idea of the micro-foundations research program – that according autonomy to participants' forecasts is the key to understanding movements of aggregate outcomes – see Frydman and Phelps (1983). For a comparison of the Phelps micro-foundations approach with recently proposed post-REH representations of participants' forecasts, see Frydman and Phelps (2013).

[10] In a notable recent book, Gennaioli and Shleifer (2018) provide a behavioral-finance account of the 2008 financial crisis, thereby advancing an argument that the profession should abandon REH models due to their inability to represent the autonomous effect of participants' forecast errors on economic outcomes.

pothesis – the core idea underpinning the REH approach – and the compelling evidence that non-fundamental factors, such as market sentiment, exert an autonomous influence on participants' forecasts, especially in asset markets.[11] As in REH models, imposing consistency within a KUH model relates participants' forecasts of aggregate outcomes to fundamentals. Remarkably, Muth's hypothesis also plays a central role in representing the influence of psychological and other non-fundamental factors on how participants' forecast outcomes *in terms of fundamentals*.[12]

In developing KUH, we build on the ideas that motivated Frydman and Goldberg's (2007, 2013a,b) attempt to formulate an approach – which they called Imperfect Knowledge Economics (IKE) – that recognizes the importance for macroeconomic theory of unforeseeable change in the process driving aggregate outcomes. Lacking the appropriate mathematical framework to characterize Knightian uncertainty in this process, Frydman and Goldberg could not rely on Muth's hypothesis to represent participants' forecasts. Consequently, they could not develop a coherent approach to building intertemporal models that recognizes that economists as well as market participants face Knightian uncertainty about the process driving outcomes. KUH offers such an approach.

Opening macroeconomics and finance models to unforeseeable change poses considerable challenges in terms of testing their predictions. The development of a methodology for testing models that recognize that econometricians, like everyone else, face Knightian uncertainty is an important topic for future research. However, in Section 9 and Appendix B, we illustrate how existing econometric methods, particularly econometrically-based calibration, as advocated by Hansen and Heckman (1996), can be adapted to confront the KUH prototype intertemporal model's predictions with time-series data on earnings, dividends, and stock prices of companies included in the S&P 500 index, as well as indicators of market sentiment extracted

[11]For an authoritative review of empirical evidence on the role of market sentiment in driving stock prices, see Barberis *et al*. (1998).

[12]We follow convention in referring to "fundamentals" as exogenous variables that an economist includes in his specifications of his model's endogenous variables. We call "non-fundamentals" the psychological as well as fundamental factors that influence only participants' forecasts of aggregate outcomes, for example the stock price and the inflation rate. For further examples and discussion, see Remark 6 in Section 8.

from the narrative market reports.[13]

The plan of the paper is as follows. Sections 2-4 explain and formally present the mathematical framework that underpins the KUH approach. We characterize Knightian uncertainty in a prototype intertemporal model and define the predictions of the model's exogenous and endogenous variables. Relying on these predictions, Sections 5-6 show how KUH applies Muth's hypothesis to represent, in terms of fundamental factors, participants' forecasts and aggregate outcomes under Knightian uncertainty. In Section 7, we show how a KUH model represents the autonomous role played by market participants' forecasts in driving outcomes. Section 8 provides two formal examples of how a KUH model represents the autonomous influence of market sentiment on participants' forecasts without presuming that participants forego profit opportunities. In Section 9, we sketch how the existing econometric methodology, including calibration, can be adapted to confront KUH models with time-series data, and we illustrate this methodology in assessing the adequacy of the predictions of a simple model for stock prices. Section 10 concludes the paper. Appendix A contains mathematical proofs of the theorems and lemmas presented in the paper. Appendix B describes the data and presents the details of our calibration methodology and econometric specifications, as well as graphs and tables of the results.

2 Characterizing Knightian Uncertainty

Macroeconomic and finance models are intertemporal in the sense that they assume that aggregate outcomes, such as the inflation rate and the stock price, are driven at each point in time by market participants' forecasts of these outcomes' future values. Regardless of the context, in order for the intertemporal representation of an aggregate outcome to generate implications for time-series data, an economist must represent participants' forecasts in terms of some exogenous variables – for example, corporate earnings in a present-value model of stock prices or total factor

[13] Shiller (2017) has argued that narrative market reports contain relevant information for building formal macroeconomics and finance models and confronting them with quantitative empirical evidence. Section 8 and Appendix B.4 illustrate the particular importance of Shiller's arguments for building mathematical models of outcomes under Knightian uncertainty, which, by definition, cannot be represented with a stochastic process.

productivity in a New Keynesian macroeconomic model.[14]

In order to present how the KUH approach formalizes both risk and Knightian uncertainty, we consider a variable, denoted by x_t, which we refer to as corporate earnings in the following. We formalize "risk" in the process driving earnings with a standard stochastic specification at a point in time. Importantly, we formalize the Knightian uncertainty faced by an economist by allowing the specification of the process driving earnings to undergo change at times and in ways that cannot be represented *ex ante* with a probabilistic rule such as Markov switching.

To focus on the key features of KUH's mathematical framework, we employ a particularly simple specification of the earnings process. We assume that log-earnings follow a random walk with time-varying drift coefficients:

$$\Delta \log x_t = \mu_t + \varepsilon_{x,t}, \tag{1}$$

for $t = 1, 2, \ldots$, and where $\{\mu_t\}_{t=1,2,\ldots}$ is a sequence of deterministic constants and $\varepsilon_{x,t}$ are independent over time with mean zero and variance σ_x^2.

The conditional moments of the probability distribution of the stochastic innovation $\varepsilon_{x,t}$, particularly its variance, represents (probabilistic) risk. Recognizing that an economist faces Knightian uncertainty about the process driving earnings, KUH does not specify a stochastic process for how the drift coefficient, μ_t, unfolds over time. Instead, KUH hypothesizes that such change can be characterized with *ex ante* conditions that constrain the values of μ_t to unfold between upper and lower bounds.[15]

Specifically, at any time t, we constrain the coefficients, $\left\{\mu_{t+k}\right\}_{k=1,2,\ldots}$, to take any value within time-varying intervals, which depend on these coefficients' values at t or earlier. We denote these intervals as follows:

$$\mu_{t+k} \in I^{\mu}_{t:t+k} = \left[L^{\mu}_{t:t+k}, U^{\mu}_{t:t+k}\right], \tag{2}$$

where $[L, U]$ indicates an interval with lower and upper bounds given by L and

[14] See Section 3, for futher discussion and references to these models.

[15] Although the KUH approach bounds the extent of change in the model's coefficients, it is compatible with large-scale unforeseeable change in how the models' variables unfold over time.

U. We write that $\mu_{t+k} \in I^{\mu}_{t:t+k}$ to indicate that μ_{t+k} lies within this interval, when viewed from time t.

We refer to the constraints, such as in (2), as Knightian uncertainty (KU) constraints. A KUH model relies on such constraints to characterize Knightian uncertainty in the processes driving its exogenous and endogenous variables. In order to explicate how the KU constraint in (2) enables us to specify such characterizations for earnings, we first note that the specification in (1) implies that earnings at time $t + k$ are given by:

$$x_{t+k} = x_t \exp(\sum_{j=1}^{k} \varepsilon_{x,t+j}) \exp(\sum_{j=1}^{k} \mu_{t+j}). \tag{3}$$

When viewed from time t, the representations in (3) specifies earnings at any future time $t + k$, x_{t+k}, in terms of (i) earnings at time t, x_t, (ii) the sequence of i.i.d. stochastic innovations, $\{\varepsilon_{x,t+j}\}$, $j = 1, 2, \ldots, k$, and, (iii) the sequence of deterministic drift coefficients $\left\{\mu_{t+j}\right\}_{j=1,2,\ldots,k}$, which undergo unforeseeable change between t and $t + k$. Thus, the probability distribution of the future x_{t+k}, conditional on the time-t realization of earnings, x_t and given the time-t values of the drift coefficient μ_t, is not defined in a KUH model.

However, the KU constraint in (2) on the future values of the drift coefficients enable us to specify at each point in time t, the probability measure, P_t, of the endpoints of the stochastic intervals within which x_{t+k} lies, conditional on x_t and given the value of μ_t. The following theorem specifies this probability measure in terms of the sequence of i.i.d. stochastic innovations, $\{\varepsilon_{x,t+j}\}_{j=1,2,\ldots,k}$:

Theorem 1 *The KU constraint in (2) on the future values of the coefficients $\left\{\mu_{t+j}\right\}$, with $1 \leq j \leq k$, implies that future earnings lie within the stochastic interval, $I^x_{t:t+k}$, when viewed from time t:*

$$x_{t+k} \in I^x_{t:t+k} = \left[L^x_{t:t+k}, U^x_{t:t+k}\right]. \tag{4}$$

where the end-points of the interval $I^x_{t:t+k}$ in (4) are given by

$$L^x_{t:t+k} = x_t \exp(\sum_{j=1}^{k} \varepsilon_{x,t+j}) \exp(\sum_{j=1}^{k} L^{\mu}_{t:t+j}), \tag{5}$$

$$U^x_{t:t+k} = x_t \exp(\sum_{j=1}^{k} \varepsilon_{x,t+j}) \exp(\sum_{j=1}^{k} U^{\mu}_{t:t+j}). \tag{6}$$

This specifies the probability distribution in terms of $\{\varepsilon_{x,t+j}\}_{j=1,2,..,k}$ conditional on x_t and for the given time-t value of μ_t.

The specification in (5)-(6) of the end-points of the stochastic intervals within which x_{t+k} lies, when viewed from time t, defines a family of the time-t conditional probability distributions, *one* of which represents earnings at $t+k$, according to the model. However, recognizing that an economist faces Knightian uncertainty, KUH does not specify at time t which of these distributions represents x_{t+k}. Because this ambiguity about the correct representation of the processes driving a KUH model's variables arises from unforeseeable change in these processes, we refer to the specifications of the stochastic intervals in (5)-(6) as a time-t Knightian uncertainty (KU) characterization of x_{t+k}.

2.1 Knightian Uncertainty Constraints

The KU characterization in (5)-(6) depends on the specifications of the KU constraints that an economist chooses *ex ante* to represent the extent of unforeseeable change in the drift coefficient $\{\mu_{t+j}\}_{j=1,2,...,k}$. As with any economic model, an economist would constrain change in a KUH model's coefficients on the basis of empirical relevance, conceptual plausibility, and tractability.

We consider a simple *ex ante* condition constraining the coefficients, μ_{t+1}, to take any value within intervals the bounds of which depend on the values of μ_t. We state this condition, which we refer to as the Knightian uncertainty (KU) constraint, as follows:

Assumption 1 *Given the value μ_t, μ_{t+1} can take any value within the interval given* *by:*

$$\mu_{t+1} \in I^{\mu}_{t:t+1} = [L^{\mu}_{t:t+1}, U^{\mu}_{t:t+1}] = [\mu_{-} + \rho_{\mu}\left(\mu_t - \mu_{-}\right), \mu_{+} + \rho_{\mu}\left(\mu_t - \mu_{+}\right)], \quad (7)$$

where $\mu_{-} < \mu_{+}$, $0 \leq \rho_{\mu} < 1$ *and the initial condition is* $\mu_{-} \leq \mu_1 \leq \mu_{+}$.

Assumption 1 neither imposes conditions on exactly how μ_t will unfold over time nor specifies a probabilistic rule to determine which value the coefficient μ_{t+1} will take within the interval $I^{\mu}_{t:t+1}$. However, the condition (7) specifies the end-points of this interval in terms of the lower and upper bounds, μ_{-} and μ_{+}, respectively, and an autoregressive parameter, ρ_{μ}.

The key implication of the KU constraint in (7) is that, when viewed from time t, Knightian uncertainty about μ_t at *any* future $t+k$ is characterized by the set of exogenously fixed constants, $(\mu_{-}, \mu_{+}, \rho_{\mu})$, which we refer to as Knightian uncertainty (KU) parameters. We state this property as a lemma:

Lemma 1 *The KU constraint (7) implies that, viewed from time* t*, for* $j \geq 1$*,*

$$\mu_{t+j} \in I^{\mu}_{t:t+j} = [L^{\mu}_{t:t+j}, U^{\mu}_{t:t+j}], \quad (8)$$

$$L^{\mu}_{t:t+j} = \mu_{-} + \rho^{j}_{\mu}\left(\mu_t - \mu_{-}\right), \quad \textit{and} \quad U^{\mu}_{t:t+j} = \mu_{+} + \rho^{j}_{\mu}\left(\mu_t - \mu_{+}\right), \quad (9)$$

and that the end-points of the intervals satisfy the following intertemporal monotonicity property:

$$L^{\mu}_{t:t+j+1} \leq L^{\mu}_{t+1:t+j+1} \quad \textit{and} \quad U^{\mu}_{t:t+j+1} \geq U^{\mu}_{t+1:t+j+1}. \quad (10)$$

Assumption 1 formalizes the idea that, when viewed from time t, Knightian uncertainty about the future values of μ_{t+j} increases with the time horizon j. That is, for $0 < \rho_{\mu} < 1$, the size of the interval $I^{\mu}_{t:t+j}$ within which μ_{t+j} lies, when viewed from time t, widens with the increase in horizon j. In this sense, Knightian uncertainty about the future values of μ_{t+j} increases with the time horizon.[16] However,

[16]For $\rho_{\mu} = 0$, the interval also reduces to $I^{\mu}_{t:t+j} = \left[\mu_{-}, \mu_{+}\right]$ for all j, thereby characterizing Knightian uncertainty about future μ_{t+j} to be the same for all time horizons j.

as $j \to \infty$, Knightian uncertainty about μ_{t+j} converges to $\left[\mu_-, \mu_+\right]$.[17]

By assuming that the evolution of the drift parameter depends on its history, and that ascertaining its range of possible values becomes increasingly difficult at a more distant horizon, the constraint in (8) provides a plausible and tractable way to characterize Knightian uncertainty in a variety of economic contexts. Because this constraint is specified in terms of a parsimonious set of parameters, a calibration methodology can be used to assess its relevance, thereby confronting a KUH model with time-series data.[18]

Section 9 presents an example of a quantitative calibration of our prototype model of the stock price on the basis of data for earnings, dividends, and stock prices of companies included in the S&P 500 Index. The results indicate that $\mu_- < 0$ and $\mu_+ > 0$, that is, μ_t could take both positive and negative values. Moreover, the empirical value of ρ_μ satisfies, $0 < \rho_\mu < 1$, which supports the idea, formalized by (8) that, viewed from time t, the extent of Knightian uncertainty increases with the time horizon j.[19]

In the next section, we show how the KU constraint in (8) underpins KUH's characterization of Knightian uncertainty in the process driving earnings.

2.2 Parametric Characterization of Knightian Uncertainty in Earnings

The moments of the stochastic innovation $\varepsilon_{x,t}$ in the specification of the earnings process in (1) characterize measurable uncertainty in this process at a point in time. The KU parameters (μ_-, μ_+, ρ_μ) play a role analogous to such moments in characterizing Knightian uncertainty in how the process driving earnings unfolds over time. The following lemma presents such a parametric characterization of KU in the earnings process:

[17]For $\rho_\mu = 1$, the constraint in (8) reduces to $\mu_{t+j} = \mu_{t+j-1}$ for all t and j, thereby assuming that an economist does not face Knightian uncertainty about the earnings process.

[18]For a pioneering argument in favor of representing outcomes in terms of a parsimonious set of exogenous parameters to facilitate the use of calibration methodology, see Prescott (1986).

[19]The econometric calibration of the model in Section 9 indicates that ρ_μ is roughly 0.7, and hence that KU increases (decreases) fast to μ_+ (μ_-).

Lemma 2 *The KU constraint in (8) on the bounds within which μ_{t+j}, $j = 1, 2, \ldots, k$, lie specifies the end-points of the stochastic interval, in (4), within which x_{t+k} lies, when viewed from time-t, in terms of x_t, μ_t and a set of exogenously fixed KU parameters (μ_-, μ_+, ρ_μ):*

$$L^x_{t:t+k} = x_t \exp(\sum_{j=1}^{k} \varepsilon_{x,t+j}) \exp(\sum_{j=1}^{k} (\mu_- + \rho_\mu^j(\mu_t - \mu_-))), \tag{11}$$

$$U^x_{t:t+k} = x_t \exp(\sum_{j=1}^{k} \varepsilon_{x,t+j}) \exp(\sum_{j=1}^{k} (\mu_+ + \rho_\mu^j(\mu_t - \mu_+))). \tag{12}$$

3 A Simple Prototype Intertemporal Model

The time-t KU characterization of exogenous variables, such as earnings, in (11) and (12), enables economists to characterize Knightian uncertainty of endogenous variables in models of a wide range of aggregate outcomes. In order to present the key features of how the KUH approach can be used to build macroeconomics and finance models, we consider a simple intertemporal model for one aggregate outcome: the stock price. The model assumes that participants bid this price to the level that satisfies the following relationship:

$$p_t = \gamma \left(\mathcal{F}_t \left(d_{t+1} \right) + \mathcal{F}_t \left(p_{t+1} \right) \right), \tag{13}$$

where p_t is the stock price, d_t denotes dividends, $\mathcal{F}_t(\cdot)$ stands for the time-t values of the market's (an aggregate of its participants') forecasts of dividends and stock prices at time $t+1$, and γ is a discount factor, which is assumed to be constant.[20]

Shiller (1981, p. 424) points out that the intertemporal representation of the stock price in (13) can be interpreted as a no-arbitrage condition, because this representation follows from equating the market's forecast of the one-period holding return from buying a stock at time t and selling it at time $t+1$ with the one-period rate of interest. He relies on (13) in his pathbreaking argument that the REH-based present-value model for the stock price is inconsistent with time-series data.[21] Al-

[20] Using a prototype model based on (13) to exposit the KUH approach is analogous to Barberis *et al.*'s (1998) reliance on a model assuming risk-neutrality and a constant discount rate to represent their approach to behavioral finance.

[21] However, the subsequent literature uncovered evidence that the discount factor is not constant,

though, for the sake of concreteness, we refer to the representation in (13) as "a no-arbitrage condition" and to the variables p_t and d_t as the "the stock price" and "dividends," our objective in this paper is not to present a fully developed KUH model of the stock price that would enable us to reexamine Shiller's findings regarding the adequacy of the present-value model under Knightian uncertainty.

We set the discount factor to a constant and make other simplifying assumptions to expound KUH's potential for building consistent intertemporal models under Knightian uncertainty and deriving their predictions for time-series data.[22] Moreover, notwithstanding the simplifying assumptions underpinning the representation of the stock price in (13), this intertemporal specification captures the key feature of the models that are typically used in other contexts in macroeconomics and finance theory. For example, analogously to the representation in (13), the New Keynesian (NK) model relates the inflation rate to participants' forecasts of its future values.[23]

4 The Knightian Uncertainty Expectation

The no-arbitrage condition (13) relates the stock price at time t to an aggregate of market participants' forecasts of dividends and stock prices in future periods. KUH relies on Muth's (1961) hypothesis to constrain the specification of participants' forecasts to be consistent with the model's prediction of these outcomes. In order to implement Muth's hypothesis in a KUH model based on the representation in (13), we must define the model's predictions of dividends and prices under Knightian uncertainty.

Because KUH opens an economist's model to unforeseeable change, the model does not represent outcomes with a stochastic process, which renders the standard (conditional) expectation undefined. Instead, we rely on KUH's characterization of Knightian uncertainty in the model's exogenous variables, such as earnings, to

leading proponents of REH to question Shiller's interpretation of his findings as a decisive rejection of the REH present-value models. For an insightful and authoritative survey of this evidence, viewed through the lens of REH models, see Cochrane (2011).

[22]Developing a KUH model of asset prices that would allow for a time-varying discount factor and relax other simplifying assumptions of the prototype model is left for future research.

[23]See Clarida *et al.* (1999) for formal examples, an extensive review of the NK models, and further references. For a classic treatment of the New Keynesian approach to monetary theory and policy, see Woodford (2003).

define the model's predictions of dividends and prices.

4.1 Characterizing Knightian Uncertainty in the Relationship Between Dividends and Earnings

As in REH models, in order to derive the model-implied relationship between the stock price and earnings, we first relate dividends to earnings. Here, we specify a linear relationship between dividends d_t and earnings x_t, according to which the impact of earnings on dividends is given by a sequence of time-varying deterministic coefficients $\{b_t\}_{t=1,2,\ldots}$,

$$d_t = b_t x_t + \varepsilon_{d,t}, \tag{14}$$

where $\varepsilon_{d,t}$ are independent over time with mean zero and variance σ_d^2. Moreover, we allow b_t to undergo unforeseeable change, but, analogously to (2), we constrain the coefficients $\{b_t\}_{t=1,2,\ldots}$, to take any value within time-varying intervals:

$$I^b_{t:t+k} = \left[L^b_{t:t+k}, U^b_{t:t+k}\right]. \tag{15}$$

Analogously to the argument leading to the Theorem 1's characterization of KU in the earnings process, the specifications in (1) and (14) imply that dividends at time $t+k$ are given by:

$$d_{t+k} = x_t b_{t+k} \exp\left(\sum_{j=1}^{k} \varepsilon_{x,t+j}\right) \exp\left(\sum_{j=1}^{k} \mu_{t+j}\right) + \varepsilon_{d,t+k}. \tag{16}$$

When viewed from time t, the representation in (16) specifies dividends at any future time $t+k$, d_{t+k}, in terms of (i) earnings at time t, x_t, (ii) the sequence of i.i.d. stochastic innovations,$\{\varepsilon_{x,t+j}\}$, $j = 1, 2, \ldots, k$, and $\varepsilon_{d,t+k}$, and, (iii) the time-varying coefficients $(\{\mu_{t+j}\}_{j=1,2,\ldots,k}, b_{t+k})$, which undergo unforeseeable change between t and $t+k$.[24] Thus, the probability distribution of the future d_{t+k}, conditional on the time-t realization of earnings, x_t and given the time-t values of the

[24] This contrasts with the KUH model's REH and behavioral-finance counterparts, which, conditional on (μ_t, b_t), would specify future values of these coefficients, $\left(\{\mu_{t+j}\}_{j=1,2,\ldots,k}, b_{t+k}\right)$ precisely – to be either unchanging over time or changing according to a probabilistic rule, such as, for example, Markov switching.

model's coefficients (μ_t, b_t), is not defined in a KUH model.

However, the KU constraints, in (2) and (15), on the future values of the model's coefficients enable us to specify at each point in time t, the probability measure, P_t, and the corresponding expectation, E_t, of the end-points of the stochastic intervals, conditional on x_t and given the values of μ_t and b_t. The following corollary to Theorem 1 specifies this probability measure in terms of the sequence of i.i.d. stochastic innovations, $\{\varepsilon_{x,t+j}\}_{j=1,2,...,k}$ and $\varepsilon_{d,t+k}$.

Corollary 1 *The KU constraints in (2) and (15) on the future values of the coefficients* $\{\mu_{t+j}\}$, $j = 1, 2, ..., k$, *and* b_{t+k} *imply that future dividends lie within the stochastic interval,* $I^d_{t:t+k}$, *when viewed from time* t, *in terms of* (μ_t, b_t),

$$d_{t+k} \in I^d_{t:t+k} = \left[L^d_{t:t+k}, U^d_{t:t+k}\right], \tag{17}$$

where the end-points of the interval $I^d_{t:t+k}$ *in (17) are given by*

$$L^d_{t:t+k} = x_t L^b_{t:t+k} \exp(\sum_{j=1}^{k} \varepsilon_{x,t+j}) \exp(\sum_{j=1}^{k} L^\mu_{t:t+j}) + \varepsilon_{d,t+k}, \tag{18}$$

$$U^d_{t:t+k} = x_t U^b_{t:t+k} \exp(\sum_{j=1}^{k} \varepsilon_{x,t+j}) \exp(\sum_{j=1}^{k} U^\mu_{t:t+j}) + \varepsilon_{d,t+k}. \tag{19}$$

This specifies the probability distribution in terms of $\{\varepsilon_{x,t+j}\}_{j=1,2,..,k}$ *conditional on* x_t *and for the given time-*t *values of* μ_t *and* b_t.

4.2 Parametric Characterization of Knightian Uncertainty in Dividends

The specifications in (18) and (19) show that the stochastic interval within which d_{t+k} lies depends on the particular specification of the KU constraint that an economist chooses *ex ante* to bound the coefficients b_{t+k} and μ_{t+k}. Although in some contexts, an economist may characterize unforeseeable change in the relationship between the two variables with different *ex ante* conditions than those characterizing such change in μ_t, here we specify the KU constraint for b_{t+k} analogously to the constraint in (7) bounding the drift of the earnings process:

Assumption 2 *Given the value b_t, b_{t+1} can take any value within the interval given by*

$$b_{t+1} \in I^b_{t:t+1} = [L^b_{t:t+1}, U^b_{t:t+1}] = [b_- + \rho_b (b_t - b_-), b_+ + \rho_b (b_t - b_+)], \quad (20)$$

where $b_- < b_+$, $0 \le \rho_b < 1$ and the initial condition is $b_- \le b_1 \le b_+$.

Analogously to Lemma 1, the following lemma specifies the KU constraint for b_{t+j}, for $j \ge 1$.

Lemma 3 *The KU constraint (20) implies that viewed from time t, for $j \ge 1$,*

$$b_{t+j} \in I^b_{t:t+j} = \left[L^b_{t:t+j}, U^b_{t:t+j}\right], \quad (21)$$

$$L^b_{t:t+j} = b_- + \rho^j_b (b_t - b_-), \quad \textit{and} \quad U^b_{t:t+j} = b_+ + \rho^j_b (b_t - b_+), \quad (22)$$

and that the end-points of the intervals satisfy the following intertemporal monotonicity property:

$$L^b_{t:t+j+1} \le L^b_{t+1:t+j+1} \quad \textit{and} \quad U^b_{t:t+j+1} \ge U^b_{t+1:t+j+1}, \quad (23)$$

where $b_+ > b_-$ and $0 \le \rho^j_b < 0$.

If $b_- > 0$, then Assumption 2 formalizes the qualitative regularity that earnings have a non-negative impact on dividends at all points in time. Although the condition (21) does not specify a particular value that b_{t+j} will take at $t + j$, this condition does constrain the value of b_{t+j} to lie within the interval, $I^b_{t:t+j}$, when viewed from time t.[25] The following lemma states the implication of this constraint for the parametric specification of Knightian uncertainty in the dividend process:

Lemma 4 *Lemmas 2 and 3 specify the end-points of the stochastic interval, in (17) within which d_{t+k} lies, when viewed from time-t, in terms of μ_t, b_t and a set of*

[25] As its counterpart for μ_{t+j} in (8), the KU constraint in (21) seems plausible in representing a time-varying relationship between dividends and earnings. In Section 9, we provide some empirical support for this condition on the basis of data for companies included in the S&P 500 stock index.

exogenously fixed parameters $(\mu_-, \mu_+, \rho_\mu, b_-, b_+, \rho_b)$:

$$L^d_{t:t+k} = x_t L^b_{t:t+k} \exp(\sum_{j=1}^{k} \varepsilon_{x,t+j}) \exp(\sum_{j=1}^{k} L^\mu_{t:t+j}) + \varepsilon_{d,t+k}, \tag{24}$$

$$U^d_{t:t+k} = x_t U^b_{t:t+k} \exp(\sum_{j=1}^{k} \varepsilon_{x,t+j}) \exp(\sum_{j=1}^{k} U^\mu_{t:t+j}) + \varepsilon_{d,t+k}, \tag{25}$$

where $L^b_{t:t+k}, U^b_{t:t+k}, L^\mu_{t:t+j},$ *and* $U^\mu_{t:t+j}$ *are given in (22) and (8) respectively.*

4.3 The Knightian Uncertainty Expectation of Dividends

Lemma 4 shows how the KU characterization of x_{t+k}, in (11) and (12) enables us to characterize the Knightian uncertainty in d_{t+k}. The model-implied representations, in (24) and (25), of the limits of the stochastic interval within which d_{t+k} lies, when viewed from time t, specify a family of the time-t conditional probability distributions. Although one of these distributions represents dividends at $t + k$, a KUH model, recognizing that an economist faces Knightian uncertainty, does not specify at time t which of them represents the process that will actually drive d_{t+k}. Thus, the (standard) time-t conditional expectation of d_{t+k} is undefined in the model.

However, the KU characterization of d_{t+k} in Lemma 4 enables us to specify the interval within which the future value of d_{t+k} is expected to lie, when viewed from time t. We refer to this interval as the Knightian uncertainty expectation (KE) of d_{t+k} and formally define this as the E_t expectation of the end-points of the interval $I^d_{t:t+k}$, in (24) and (25), within which d_{t+k} is expected to lie, conditional on x_t and for the given values of μ_t and b_t:

$$KE_t\left(d_{t+k}\right) = \left[E_t\left(L^d_{t:t+k}\right), E_t\left(U^d_{t:t+k}\right)\right]. \tag{26}$$

Computing E_t of the end-points in (24) and (25) we find that,

$$KE_t\left(d_{t+k}\right) = x_t v^k \left[l^d_{t:t+k}, u^d_{t:t+k}\right], \tag{27}$$

where $v = E\exp(\varepsilon_{x,t})$ and

$$l^d_{t:t+k} = \left(b_- + \rho_b^k\left(b_t - b_-\right)\right)\exp(\sum_{j=1}^{k}(\mu_- + \rho_\mu^j(\mu_t - \mu_-))), \tag{28}$$

$$u^d_{t:t+k} = \left(b_+ + \rho_b^k\left(b_t - b_+\right)\right)\exp(\sum_{j=1}^{k}(\mu_+ + \rho_\mu^j(\mu_t - \mu_+))). \tag{29}$$

The interval $KE_t\left(d_{t+k}\right)$ in (27) represents the time-t prediction of the range of values within which dividends are expected to lie at $t+k$ in terms of $x_t, \mu_t, b_{t,}$, the set of exogenously fixed KU parameters, $(\mu_-, \mu_+, \rho_\mu, b_-, b_+, \rho_b)$, and the moments of the innovation, $\varepsilon_{x,t}$.

Note that if we consider the conditional expectation of d_t given x_t there is no role for Knightian uncertainty in how dividends unfold over time. But we can formally define the Knightian expectation as the point (a degenerate interval) given by,

$$KE_t\left(d_t\right) = E_t\left(d_t\right) = b_t x_t.$$

4.3.1 Iterated Knightian Uncertainty Expectations

The analysis of the implications of the no-arbitrage condition in (13), involves iterations of KE, such as $KE_t\left(KE_{t+1}(d_{t+2})\right)$. This involves two (or more) iterations. First, as discussed above, $KE_{t+1}(d_{t+2})$ is computed as E_{t+1} of the end-points of the stochastic interval implied by the KU constraints for d_{t+2}, when viewed from time $t+1$. The end-points of the resulting interval, $KE_{t+1}(d_{t+2})$, depend on the time $t+1$ values of b_{t+1} and μ_{t+1}, as well as earnings, x_{t+1}; thus, the conditional expectation E_t of these end-points is not well defined at time t. However, applying the KU constraints again, as well as iterating x_{t+1}, enables us to express these end-points in terms of b_t and μ_t (and x_t). This renders E_t of the end-points of $KE_{t+1}(d_{t+2})$, and thus $KE_t\left(KE_{t+1}(d_{t+2})\right)$, well-defined. The following theorem derives such iterated Knightian expectations and shows that the constraints for b_{t+j} and μ_{t+j}, in (22) and (9), imply that the analog of the law of iterated expectations holds under Knightian uncertainty expectations.

Theorem 2 *With $KE_t\left(d_{t+k}\right)$ defined in (26), it follows for $k \geq 0$ under the KU* *constraints in (8) and (21) that:*

$$KE_t\left(d_t\right) = b_t x_t, \tag{30}$$

$$KE_t\left(d_{t+k}\right) = x_t[L^b_{t:t+k} v^k \exp(\sum_{j=1}^{k} L^{\mu}_{t:t+j}), U^b_{t:t+k} v^k \exp(\sum_{j=1}^{k} U^{\mu}_{t:t+j})], \tag{31}$$

where $v = E\exp\left(\varepsilon_{x,t}\right)$*, and* $L^b_{t:t+k}$*,* $U^b_{t:t+k}$*,* $L^{\mu}_{t:t+j}$*, and* $U^{\mu}_{t:t+j}$ *are specified in (22) and (9). Furthermore, the following iterative property of KE holds:*

$$KE_t\left(d_{t+k}\right) = KE_t\left(KE_{t+1}\left(\ldots KE_{t+k-1}\left(d_{t+k}\right)\ldots\right)\right). \tag{32}$$

Remark 1 *The property in (32) may be viewed as a law of iterated Knightian uncertainty expectations. It holds for the characterizations of Knightian uncertainty in the dividends process in (24)-(25).*[26]

5 Muth's Hypothesis Under Knightian Uncertainty

KUH's representations of market participants' forecasts of outcomes, such as dividends and stock prices, rest on the premise that participants are rational, in the sense that they are goal-oriented (typically assumed to mean profit-seeking) and relate the forecasts of payoff-relevant outcomes to some understanding, albeit imperfect, of the process driving these outcomes. Muth (1961) argued that an economist can relate participants' forecasts to rational considerations by representing their understandings of the processes driving dividends and stock prices with his own understanding of these processes, as formalized by his model.

Muth (1961, pp. 315-317) was well aware that "there are considerable [...] differences of opinion" about the processes driving outcomes. Importantly, he emphasized that his hypothesis should not be "confused [...] with a pronouncement as to what [rational participants] ought to do," and that it does not assert that their forecasts, are "perfect." Muth believed that, although boldly abstract, representing participants' forecasts as being consistent with models' predictions of outcomes

[26]This property may, or may not, hold for a different KU characterizations. This is in contrast to REH models where the complete stochastic specifications for all variables implies that the law of iterated expectations always applies. We leave for future research the characterization of a general class of KU constraints under which the property in (32) holds.

was a "sensible" way to acknowledge participants' rationality – that their forecasts are related to "the way the economy works." That, after all, is precisely what an economist hypothesizes and formalizes with his own model.

Relying on this premise, Lucas (1995, p. 254-255) argued that Muth's hypothesis should be considered "the principle" of coherent model building in macroeconomic and finance theory. He pointed out that when an economist relates participants' forecasts to how "the economy works" in a way that is inconsistent with the predictions of his own model, he contradicts his model's hypothesis: that it represents how outcomes actually unfold over time.

By imposing consistency within an intertemporal model, REH removed the "glaring" inconsistency that characterized the intertemporal macroeconomic models of the 1960s. Analogously, KUH relies on Muth's hypothesis to construct coherent models that recognize that not only market participants, *but economists as well*, face Knightian uncertainty about the process driving outcomes.

5.1 REH's Implementation of Muth's Hypothesis

In order to highlight the main distinctive features of KUH's application of Muth's hypothesis, we first briefly consider REH's application of the hypothesis in the context of the specifications of earnings and dividends in (1) and (14).

Constraining $\mu_+ = \mu_- = \mu$ and $b_+ = b_- = b$, in (7) and (20), formalizes the assumption that the processes driving earnings and dividends do not undergo unforeseeable change, thereby reducing our KUH prototype to its REH counterpart.

As we next illustrate formally, REH's application of Muth's hypothesis about how economists' models can recognize that participants rely on rational considerations has a crucial implication: the conditional expectation of an economist's own stochastic specification of dividends represents *precisely* how *every* market participant understands and forecasts dividends.

To this end, we let $\mathcal{F}_t^i(d_{t+k})$ and $\mathcal{F}_t^j(d_{t+k})$ denote the values of the forecasts of d_{t+k} selected by any two market participants, say i and j. The REH version of our prototype represents the time-t forecasts of dividends by every participant, as well as the market, at any $t + k$ for $k \geq 1$ and for *any* (participants) i and j as follows:

$$\mathcal{F}_t^i(d_{t+k}) = \mathcal{F}_t^j(d_{t+k}) = \mathcal{F}_t(d_{t+k}) = \varphi x_t, \tag{33}$$

where $\mathcal{F}_t(d_{t+k})$ denotes the value of the market's (an aggregate of its participants) forecast, and

$$\varphi = v^k b \exp(k\mu). \tag{34}$$

Remark 2 *The representation in (34) illustrates the key implication of assuming that the process driving outcomes, such as dividends, does not undergo unforeseeable change. Applying Muth's hypothesis in such models, as REH does, constrains representations of participants' forecasts of dividends at each $t + k$, d_{t+k}, to be uniform, in the sense that every market participant is assumed to select exactly the same quantitative forecast of dividends in making his demand and supply decisions. Moreover, REH fully determines the* singular *representation of the so-called "representative agent's" forecasts in terms of the model's coefficients, (b, μ) and the moments of its stochastic innovations, v.*

5.2 Relating Participants' Forecasts of Dividends to Earnings under Knightian Uncertainty

Having defined an economic model's predictions under Knightian uncertainty, we apply Muth's hypothesis in our prototype KUH model. As with REH models, the hypothesis enables an economist using a KUH model to acknowledge rationality in participants' forecasting, thereby relating the specification of their forecasts of aggregate outcomes, such as the stock price, to fundamental factors, such as earnings. However, in contrast to REH's representations, imposing consistency in a model that recognizes an economist's Knightian uncertainty yields neither precise nor uniform representations of participants' forecasts.

A KUH model formalizes an economist's understanding that the process driving outcomes undergoes unforeseeable change. KUH implements Muth's hypothesis by constraining the model's representations of participants' forecasts of dividends and stock prices to be consistent with its predictions of these outcomes. Consequently, the model does not represent how participants forecast these outcomes with a stochastic process. Applying Muth's hypothesis in a KUH model thus represents that market participants also understand that the process driving outcomes undergoes unforeseeable change.

To demonstrate this formally, we show how our KUH prototype represents participants' forecasts of dividends in terms of earnings. The KE expectation in (31) specifies the interval within which d_{t+k} is expected to lie, according to the KU characterization of dividends in (24)-(25). Applying Muth's hypothesis, we represent the value of the ith participant's time-t forecast of d_{t+k} to be one of the points within the KE interval in (31):

$$\mathcal{F}_t^i(d_{t+k}) = \varphi_{t:t+k}^i x_t \in KE_t\left(d_{t+k}\right). \tag{35}$$

The expression for $KE_t\left(d_{t+k}\right)$ in (31) implies that, according to the model, the interval, $I_{t:t+k}^{\varphi}$, within which φ^i lies is given by

$$\varphi_{t:t+k}^i \in I_{t:t+k}^{\varphi} = \left[L_{t:t+k}^{\varphi}, U_{t:t+k}^{\varphi}\right] = v^k \left[l_{t:t+k}^d, u_{t:t+k}^d\right], \tag{36}$$

where $l_{t:t+k}^d$ and $u_{t:t+k}^d$ are specified in (28) and (29).

The representation in (35)-(36) formalizes the idea that recognizing that an economist faces Knightian uncertainty means that his model does not determine completely which particular value of $\mathcal{F}_t^i(d_{t+k})$ a market participant will select at time t.[27] However, although Muth's hypothesis neither determines the particular values that the coefficients $\varphi_{t:t+k}^i$ in (36) take for any i, nor restricts these coefficients to be the same for all i, the hypothesis does constrain the values of all $\varphi_{t:t+k}^i$s to lie within the interval $I_{t:t+k}^{\varphi}$, in (36). Denoting an aggregate of $\varphi_{t:t+k}^i$s by $\varphi_{t:t+k}$, and the corresponding aggregate of $\mathcal{F}_t^i\left(d_{t+1}\right)$ by $\mathcal{F}_t\left(d_{t+1}\right)$), we formally state this:

$$\mathcal{F}_t\left(d_{t+k}\right) = \varphi_{t:t+k} x_t, \tag{37}$$

[27] If an economist were to specify a probabilistic rule that would enable him to represent which specific value $\mathcal{F}_t^i(d_{t+k})$ takes within the interval $KE_t\left(d_{t+k}\right)$, in (35), he would be able to represent aggregate outcomes with a stochastic process, thereby assuming that he does not face Knightian uncertainty. For an example of such an approach and further discussion, see Ilut and Scheider (2014).

where $\varphi_{t:t+k} \in I^{\varphi}_{t:t+k} = \left[L^{\varphi}_{t:t+k}, U^{\varphi}_{t:t+k}\right]$, and the model does not specify the particular value that $\varphi_{t:t+k}$ takes within the interval $I^{\varphi}_{t:t+k}$.

Herein lies the true significance of Muth's hypothesis for macroeconomics and finance theory, as well as macroeconometrics under Knightian uncertainty. Once an economist recognizes that the process driving aggregate outcomes undergoes unforeseeable change, he faces ambiguity about the precise values of the forecasts that underpin rational participants' decisions. However, because KUH characterizes Knightian uncertainty with ex ante constraints on the extent of unforeseeable change, applying Muth's hypothesis in the model enables an economist to impose bounds on his ambiguity about these values.

As we show in the remainder of this paper, such bounds on representations of participants' forecasts of dividends are essential to a KUH model's derivation of the relationship between the stock price and earnings. Moreover, because Muth's hypothesis does not constrain a KUH model's representation of participants' forecasts fully, the hypothesis enables economists to represent the roles played by both fundamental and non-fundamental factors, such as market sentiment, in how participants forecast dividends and stock prices. KUH thus reveals a path to building macroeconomics and finance models that synthesize the core ideas underpinning the REH and behavioral-finance approaches in a way that is compatible with the diversity and rationality that characterize participants' forecasts.

6 The Stock Price under Knightian Uncertainty

In Section 5.1, we illustrated how the REH counterpart of our KUH prototype determines the particular values of the model-consistent representation of $\mathcal{F}_t\left(d_{t+1}\right)$. As is well known, applying REH in the intertemporal representation in (13),

$$p_t = \gamma\left(\mathcal{F}_t\left(d_{t+1}\right) + \mathcal{F}_t\left(p_{t+1}\right)\right), \tag{38}$$

determines the particular values of the stock price set by the market at time t, p_t in terms of the REH model's coefficients and the moments of its stochastic innovations.

Like its REH counterpart, a KUH model assumes that the no-arbitrage condition in (38) summarizes how market participants' demand and supply decisions – made on the basis of the *specific* values of their forecasts of dividends and prices, as aggregated by $\mathcal{F}_t(d_{t+1})$ and $\mathcal{F}_t(p_{t+1})$ – set the value of p_t. However, recognizing Knightian uncertainty on the part of an economist, a KUH model does not specify the particular value of the market's quantitative forecast, $\mathcal{F}_t(d_{t+1})$. Instead, from (37), the model represents this forecast to lie in the interval, that is, $\mathcal{F}_t(d_{t+1}) \in KE_t(d_{t+1})$. Consequently, the KUH model does not represent the particular value of the stock price. However, as we show next, the model specifies the interval within which the value of the price p_t set by the market, according to (38) lies at each t.

6.1 A No-Arbitrage Price Interval

In this section, we define the concept of a no-arbitrage interval $\mathcal{I}_t^p$ that satisfies the interval analog of the no-arbitrage condition in (38). Moreover, we show that at any point in time, $p_t \in \mathcal{I}_t^p$, while $\mathcal{F}_t(d_{t+1}) \in KE_t(d_{t+1})$ and $\mathcal{F}_t(p_{t+1}) \in KE_t\left(\mathcal{I}_{t+1}^p\right)$.

In order to define $\mathcal{I}_t^p$, we make the following assumption, which is the interval version of the well-known transversality condition.

Assumption 3 *Assume that γ is chosen such that $\gamma v \exp\left(\mu_+\right) < 1$, where $v = E\exp(\varepsilon_{x,t})$.*

Given this assumption, we define $\mathcal{I}_t^p$ as follows.

Definition 1 *Using the representation of the KE intervals in (31) within which d_{t+k} is expected to lie, we define the following interval:*

$$\mathcal{I}_t^p = \sum_{k=1}^{\infty} \gamma^k KE_t(d_{t+k}). \tag{39}$$

Because, as the following theorem establishes, $p_t \in \mathcal{I}_t^p$ and $\mathcal{I}_t^p$ satisfies the interval analogue of the (pointwise) no-arbitrage condition in (38), we refer to $\mathcal{I}_t^p$ as a no-arbitrage price interval.

Theorem 3 *Under Assumption 3, the interval $\mathcal{I}_t^p$ in (39) is well-defined, and satisfies the no-arbitrage interval condition,*

$$\mathcal{I}_t^p = \gamma\left(KE_t(d_{t+1}) + KE_t\left(\mathcal{I}_{t+1}^p\right)\right). \tag{40}$$

Moreover, $\mathcal{I}_t^p$ is given by,

$$\mathcal{I}_t^p = \sum_{k=1}^{\infty} \gamma^k K E_t \left(d_{t+k}\right) = x_t \left[L_t^p, U_t^p\right], \tag{41}$$

where

$$L_t^p = \sum_{k=1}^{\infty} \gamma^k v^k L_{t:t+k}^b \exp\left(\sum_{j=1}^{k} L_{t:t+j}^{\mu}\right), \tag{42}$$

$$U_t^p = \sum_{k=1}^{\infty} \gamma^k v^k U_{t:t+k}^b \exp\left(\sum_{j=1}^{k} U_{t:t+j}^{\mu}\right), \tag{43}$$

and the model-implied bounds $L_{t:t+k}^b$, $U_{t:t+k}^b$, $L_{t:t+j}^{\mu}$, and $U_{t:t+j}^{\mu}$ are specified in (9) and (22).

Remark 3 *In the special case in which $\rho_\mu = \rho_b = 0$, in (9) and (22), the Knightian uncertainty about future b_{t+k} and μ_{t+k} is the same for all horizons, k, and the no-arbitrage stock-price interval in (39) simplifies to $\mathcal{I}_t^p = x_t \left[L^p, U^p\right]$, with*

$$L^p = b_- \tfrac{\gamma v \exp(\mu_-)}{1-\gamma v \exp(\mu_-)} \quad \textit{and} \quad U^p = b_+ \tfrac{\gamma v \exp(\mu_+)}{1-\gamma v \exp(\mu_+)}.$$

Remark 4 *Constraining $\mu_+ = \mu_- = \mu$ and $b_+ = b_- = b$, reduces our KUH prototype to its REH counterpart. This collapses the no-arbitrage stock-price interval in Theorem 3 to the point that represents the value of the stock price set by the market precisely and determines it completely in terms of the model's coefficients and the moments of its stochastic innovations:*

$$p_t = \theta x_t, \quad \textit{with} \quad \theta = b \tfrac{\gamma v \exp(\mu)}{1-\gamma v \exp(\mu)}. \tag{44}$$

Like its REH counterpart in (44) a KUH model's representation in (41) relates stock prices to earnings, which we may write as

$$p_t = \theta_t x_t \ \in \mathcal{I}_t^p, \tag{45}$$

where, with L_t^p and U_t^p defined in (42) and (43), $\theta_t \in \left[L_t^p, U_t^p\right]$ such that θ_t is a time-varying coefficient. However, in contrast to its REH counterpart's representation for p_t in (44), the KUH model does not imply the specific value of θ_t and thus p_t, at any t.

Remark 5 *Although a KUH model does not specify the particular values that θ_t, and thus p_t, actually take within their respective intervals, the model-implied specifications of the intervals $KE_t\left(d_{t+k}\right)$, in (37), mean that p_t lies within the interval $\mathcal{I}_t^p$ in (41). From (9) and (22), the end-points of this interval, in (42) and (43), depend on x_t, μ_t, b_t, the set of exogenously fixed KU parameters, $(\mu_-, \mu_+, \rho_\mu, b_-, b_+, \rho_b)$, and the moments of the innovation, $\varepsilon_{x,t}$.*[28]

6.2 Relating Participants' Forecasts of Stock Prices to Earnings Under Knightian Uncertainty

As we did in representing participants' forecasts of dividends, we rely on Muth's hypothesis to constrain the model's representation of $\mathcal{F}_t\left(p_{t+1}\right)$. To this end, we note that the specification of the no-arbitrage interval in $\mathcal{F}_t\left(p_{t+1}\right)$ implies that $\mathcal{I}_{t+1}^p$ depends on x_{t+1}, the values of which are expected to lie within the interval (4) for $k=1$, when viewed from time t.Thus, the time-t Knightian uncertainty expectation of $\mathcal{I}_{t+1}^p$ is given by:

$$KE_t\left(\mathcal{I}_{t+1}^p\right) = x_t[L_t^\phi, U_t^\phi], \tag{46}$$

where

$$L_t^\phi = \sum_{k=1}^{\infty} \gamma^k v^{k+1} L_{t:t+1+k}^b \exp\left(\sum_{j=1}^{k+1} L_{t:t+j}^\mu\right), \tag{47}$$

$$U_t^\phi = \sum_{k=1}^{\infty} \gamma^k v^{k+1} U_{t:t+1+k}^b \exp\left(\sum_{j=1}^{k+1} U_{t:t+j}^\mu\right). \tag{48}$$

Applying Muth's hypothesis, the KUH model represents an ith participant's forecast, $\mathcal{F}_t^i\left(p_{t+1}\right)$, to be a point in the interval $KE_t\left(\mathcal{I}_{t+1}^p\right)$ within which the price set

[28] Such model-implied parameterization of the interval within which the price actually set by the market at time t lies enables us to confront the interval predictions of a KUH model with econometric estimates of the model's time-varying coefficients and the moments of the model's stochastic innovations. In Section 9 and Appendix B.3, we illustrate such a quantitative calibration exercise, which combines Kydland and Prescott's (1982) original calibration methodology with an explicit econometric framework, as advocated by Hansen and Heckman (1996).

by the market at $t+1$ is expected at time t to lie:

$$\mathcal{F}_t^i\left(p_{t+1}\right) \in KE_t\left(\mathcal{I}_{t+1}^p\right).$$

Denoting by $\mathcal{F}_t\left(p_{t+1}\right)$ the market's (aggregate) forecast of p_{t+1}, the expressions for $KE_t\left(p_{t+1}\right)$ in (46)-(48) imply that, according to the model,

$$\mathcal{F}_t\left(p_{t+1}\right) = \phi_t x_t, \tag{49}$$

where $\phi_t \in I_t^{\phi} = [L_t^{\phi}, U_t^{\phi}]$.

7 How Fundamentals Drive Stock Prices: An Autonomous Role for Participants' Forecasts

We have shown that imposing consistency within a KUH model, in contrast to doing so within its REH counterpart, does not fully constrain representations of participants' forecasts of dividends and prices. In this sense, a KUH model formalizes the idea that participants' forecasts play an *autonomous* role in driving aggregate outcomes, such as the stock price. To present this point formally, we note that by Theorem 3, the no-arbitrage condition in (38), implies that

$$p_t \in \mathcal{I}_t^p = \gamma(KE_t\left(d_{t+1}\right) + KE_t\left(\mathcal{I}_{t+1}^p\right)). \tag{50}$$

Thus, the model represents the price actually set by the market to be one of the points within the interval $\mathcal{I}_t^p$. Moreover, although the model does not specify the particular value that p_t takes, it does assume that this price satisfies the intertemporal relationship $p_t = \gamma\left(\mathcal{F}_t\left(d_{t+1}\right) + \mathcal{F}_t\left(p_{t+1}\right)\right)$. Using the representation of $\mathcal{F}_t\left(d_{t+1}\right)$, in (37) for $k = 1$, and the representation of $\mathcal{F}_t\left(p_{t+1}\right)$, in (49), we can formally write p_t as follows:

$$p_t = \gamma(\varphi_t + \phi_t)x_t, \tag{51}$$

where $\varphi_t \in I_{t:t+1}^{\varphi} = \left[L_{t:t+1}^{\varphi}, U_{t:t+1}^{\varphi}\right]$ and $\phi_t \in I_t^{\phi} = \left[L_t^{\phi}, U_t^{\phi}\right]$, while $L_{t:t+1}^{\varphi}, U_{t:t+1}^{\varphi}, L_t^{\phi}$ and U_t^{ϕ} are specified in (36) and in (47)-(48). Also to simplify notation, φ_t stands for $\varphi_{t:t+1}$.

Recognizing that an economist's faces Knightian uncertainty, KUH does not represent the precise values that φ_t and ϕ_t, and thus $\mathcal{F}_t(d_{t+1})$ and $\mathcal{F}_t(p_{t+1})$, take within their respective intervals. Thereby, KUH implies that an economist faces *ambiguity* about how rational participants forecasts drive their demand and supply decisions, which, in turn, result in the price p_t set by the market at t. This ambiguity – that an economist's model does not fully constrain its representations of participants' forecasts – is just another way of stating that these forecasts play an autonomous role in how the model represents the price at t, p_t.

The autonomy of participants' forecasts in setting the stock price at a point in time implies that participants' revisions of their forecasts play an autonomous role in driving the movements of aggregate outcomes, such as the stock price, over time. Using (51), we state this formally as follows using (51):

$$\Delta p_t = \gamma\left(\varphi_t + \phi_t\right)\Delta x_t + \gamma\left(\Delta\varphi_t + \Delta\phi_t\right)x_{t-1}, \tag{52}$$

where $\Delta p_t = p_t - p_{t-1}$. The term $\gamma\left(\varphi_t + \phi_t\right)\Delta x_t$ represents the effect of the change in earnings on stock prices between $t-1$ and t, while the term $\gamma\left(\Delta\varphi_t + \Delta\phi_t\right)x_{t-1}$ represents the effect of the change in the coefficients, which may depend on other factors (for example, market sentiment) on how participants forecast dividends and prices between these periods as illustrated in the next sections.

8 Reconciling Model Consistency with Behavioral Evidence

Imposing consistency within a KUH model relates participants' forecasts of aggregate outcomes to fundamentals. By reconciling Muth's hypothesis with the autonomy of participants' forecasts, KUH opens a way to build macroeconomic and finance models that represent the influence of psychological and other non-fundamental considerations on how participants forecast outcomes in terms of fundamentals, and thus aggregate outcomes, without presuming that participants are irrational.

Remark 6 *As we mentioned in the Introduction, we follow the convention in referring to "fundamental" and "non-fundamental" factors as exogenous variables that, respectively, an economist includes and does not include in his specification of div-*

idends, in (14). Thus, our KUH prototype includes only one fundamental factor: corporate earnings. Although we refer to all other factors as non-fundamentals, they can include both psychological considerations and other factors, such as aggregate economic activity or sales, which influence participants' forecasts of dividends and stock prices directly (rather than through their effect on dividends, as specified by the model).

8.1 Participants' Reliance on Non-Fundamental Factors as an Implication of Model Consistency

A KUH model implies that, like the economist, market participants also face Knightian uncertainty. Thus, the model implies on *logical grounds* that a profit-seeking participant understands that any statistical model at best approximates some incomplete aspect of the process driving outcomes. Moreover, unforeseeable change may occur at any point in time, thereby rendering any stochastic approximation of the process underpinning outcomes inconsistent with how outcomes actually unfold. Thus, a KUH model's consistency implies that a market participant faces ambiguity about which stochastic specification approximates the past relationships between dividends, prices, and earnings, let alone which specifications might approximate these relationships in the future.

Consequently, in selecting a particular quantitative forecast to underpin his demand and supply decisions, a *rational* market participant relies on a variety of factors and methods. These include the predictions of a multitude of economic models on offer, as well as assessments of the effects of psychological and other non-fundamental considerations, such as market sentiment or political events, on the future course of aggregate outcomes.

8.2 Formal Representation of Market Sentiment

By constraining the model-consistent coefficients of representations of participants' forecasts *only* partly, a KUH model formalizes the relevance of myriad factors that drive participants' forecasts but that an economist cannot build into his necessarily parsimonious and abstract model. In this section, we sketch two examples of how our KUH prototype can represent the autonomous influence of non-fundamental factors on participants' forecasts of dividends and stock prices with mathematical conditions that constrain the model-consistent intervals of the model's representations of these forecasts.

To this end, we define an exogenous variable, s_t, which, for the sake of concreteness, we refer to as an aggregate of market participants' sentiment about the future course of corporate earnings, dividends, and/or stock prices. We specify s_t to take three discrete values, which we refer to as the state of this market sentiment:

$$s_t = \begin{cases} 1, \text{if the market (sentiment) is optimistic,} \\ 0, \text{if the market (sentiment) is neutral,} \\ -1, \text{if the market (sentiment) is pessimistic.} \end{cases} \tag{53}$$

Remark 7 *In an economic model, market sentiment affects the representation of participants* quantitative *forecasts of outcomes. However, the "sentiment" itself stands for the influence of a variety of factors, including psychological, political, and other* qualitative *as well as quantitative factors that* participants consider relevant, *but that an economist has not included in his mathematical model.*[29]

8.3 Behavioral Finance: Representing the Role of Market Sentiment with Inconsistent Models

Behavioral-finance theorists have amassed compelling empirical evidence that market sentiment has significant autonomous effect on aggregate outcomes, especially in asset markets. However, they often represent their findings with models that specify with a stochastic process how outcomes unfold over time, thus not including unforeseeable change. This led them to rely on inconsistent models.

We illustrate such behavioral-finance representations by constraining the coefficients μ_t and b_t in (1) and (14) to be equal to constants, μ and b, thereby reducing representations of $\mathcal{F}_t\left(d_{t+1}\right)$ and $\mathcal{F}_t\left(p_{t+1}\right)$ in our prototype to their REH counter-

[29] Sentiment measures are constructed on the basis of narrative reports covering various historical events and market participants' interpretations of them – specifically, whether participants consider these events positive (negative) for the future course of the economy. See Section 9 and Appendix B.4 for the use of such measures in confronting our prototype model with time-series data and further references.

parts:

$$\mathcal{F}_t(d_{t+1}) = \varphi^{reh} x_t, \text{ and } \mathcal{F}_t(p_{t+1}) = \phi^{reh} x_t, \tag{54}$$

where, from (1), (33), and (44):

$$\varphi^{reh} = vb\exp(\mu), \quad \text{and} \quad \phi^{reh} = b\tfrac{\gamma(v\exp(\mu))^2}{1-\gamma v\exp(\mu)}. \tag{55}$$

Having represented their empirical findings with models that rule out unforeseeable change in how market sentiment influences participants' forecasts, behavioral-finance theorists have had no option but to represent these forecasts with inconsistent models. A particularly simple example of such a representation hypothesizes that, conditional on the time-t earnings, x_t, when the market is optimistic (pessimistic) its forecasts of dividends and prices exceed (fall short of) their REH-implied values. This behavioral hypothesis can be formally specified as constraints on participants' forecasts of dividends and prices, such that when the market is optimistic, $s_t = 1$:

$$\mathcal{F}_t(d_{t+1}) = \varphi^{opt} x_t, \text{ and } \mathcal{F}_t(p_{t+1}) = \phi^{opt} x_t, \tag{56}$$

where $\varphi^{opt} > \varphi^{reh}$ and $\phi^{opt} > \phi^{reh}$. Likewise, for the case of pessimisim, $s_t = -1$:

$$\mathcal{F}_t(d_{t+1}) = \varphi^{pes} x_t, \text{ and } \mathcal{F}_t(p_{t+1}) = \phi^{pes} x_t, \tag{57}$$

where $\varphi^{pes} < \varphi^{reh}$ and $\phi^{pes} < \phi^{reh}$, with $\left(\varphi^{opt}, \phi^{opt}\right)$ and $\left(\varphi^{pes}, \phi^{pes}\right)$ constants.

Remark 8 *Constraints in (56) and (57) illustrate the key features of behavioral-finance representations of the influence of market sentiment (as well as other non-fundamental factors) on participants' forecasts:*

1. *These representations are necessarily inconsistent with how outcomes such as dividends or prices actually unfold over time, as hypothesized by an economist's model.*
2. *They assume that market sentiment has the same influence on every participant's forecasts and that this effect is either unchanging over time or can be*

represented with a probabilistic rule, such as Markov switching.[30]

Thus the representations in (56) and (57) presume that when participants are optimistic (pessimistic), they necessarily forego profit opportunities. In the next section, we formulate a KUH analog of the behavioral-finance representation in (56) and (57). We show how recognizing that an economist faces Knightian uncertainty enables him to represent the *diverse*, *autonomous* influences of market sentiment on participants' forecasts, and thus on stock prices, in a way that does not presume that, when they are optimistic (pessimistic) they forego profit opportunities.

8.4 KUH: Representing the Role of Market Sentiment in Consistent Models

As we summarized in Remark 8, behavioral-finance models suffer from theoretical and (likely) empirical difficulties, owing to their assumption that neither economists nor market participants face Knightian uncertainty. However, the idea underpinning Barberis *et al*.'s. (1998) behavioral-finance constraints in (56) and (57) – that optimism (pessimism) leads participants to select forecasts that tend to be higher (lower) than those chosen when the market is in a neutral state – nonetheless seems a sensible way to represent the influence of market sentiment on participants' forecasts.

However, in contrast to REH models, applying Muth's hypothesis in a KUH model does not represent participants' forecasts of dividends and prices with precise values. This opens a way to represent the influence of market sentiment (and other non-fundamental factors) on participants' forecasts as a constraint on the model-consistent consistent KE intervals within which participants' forecasts lie, according to the model.

There are a number of ways to formulate an analog of the constraints in (56) and (57) in a KUH model. Here, we present two examples of such representations.[31]

[30]These features characterize a seminal behavioral-finance model of the role of market sentiment in driving stock prices by Barberis *et al*. (1998). They formulate the model for a "representative investor" whose forecasts switch between two models of earnings, each inconsistent with an economist specification, according to a Markov switching rule.

[31]A more complete exposition of how KUH can represent the insights of behavioral finance in consistent models, that is, without presuming that market participants are irrational, is beyond the scope of this paper.

8.4.1 Modifying Bounds for Representations of Participants' Forecasts

We state the representations of optimism and pessimism as the following hypothesis:

Hypothesis 1

(i) *When the market is optimistic, its forecasts lie within the following upper subintervals of the model-consistent KE intervals, $KE_t(d_{t+1})$ in (31) and $KE_t(\mathcal{I}^p_{t+1})$ in (46):*

$$\mathcal{F}_t^{opt}(d_{t+1}) \in I^{\varphi}_{t:t+1}(s_t = 1) = \left[L^{\varphi}_{t:t+1}(s_t = 1), U^{\varphi}_{t:t+1}\right], \tag{58}$$

$$\mathcal{F}_t^{opt}(p_{t+1}) \in I^{\phi}_{t}(s_t = 1) = [L^{\phi}_{t}(s_t = 1), U^{\phi}_{t}], \tag{59}$$

where

$$L^{\varphi}_{t:t+1}(s_t = 1) = L^{\varphi}_{t:t+1}(1-\eta) + U^{\varphi}_{t:t+1}\eta, \tag{60}$$

$$L^{\phi}_{t}(s_t = 1) = L^{\phi}_{t}(1-\eta) + U^{\phi}_{t}\eta, \tag{61}$$

and $0 \leq \eta \leq 1$. We note that as η increases, the lower end-points of the restricted intervals, $L^{\varphi}_{t:t+1}(s_t = 1)$ and $L^{\phi}_{t}(s_t = 1)$ increase. We therefore refer to η as the sentiment effect.

(ii) When the market is pessimistic, its forecasts lie within the following lower subintervals of the model-consistent KE intervals, $KE_t(d_{t+1})$ in (31) and $KE_t(\mathcal{I}^p_{t+1})$ in (46):

$$\mathcal{F}_t^{pes}(d_{t+1}) \in I^{\varphi}_{t:t+1}(s_t = -1) = \left[L^{\varphi}_{t:t+1}, U^{\varphi}_{t:t+1}(s_t = -1)\right], \tag{62}$$

$$\mathcal{F}_t^{pes}(p_{t+1}) \in I^{\phi}_{t}(s_t = -1) = \left[L^{\phi}_{t}, U^{\phi}_{t}(s_t = -1)\right], \tag{63}$$

where

$$U^{\varphi}_{t:t+1}(s_t = -1) = U^{\varphi}_{t:t+1}(1-\eta) + L^{\varphi}_{t:t+1}\eta, \tag{64}$$

$$U^{\phi}_{t}(s_t = -1) = U^{\phi}_{t}(1-\eta) + L^{\phi}_{t}\eta, \tag{65}$$

and $0 \leq \eta \leq 1$.

Remark 9 *The constraints representing the effect of optimism and pessimism on participants' forecasts, in Hypothesis 1 (i) and (ii), highlight how recognizing that an economist faces Knightian uncertainty enables him to remedy the difficulties inherent in the behavioral-finance formalizations of market sentiment in (56) and (57). In contrast to Remark 8:*

(i) *By design, the representations implied by KU constraints in (58)-(59) and (62)-(63) are model-consistent, thus avoiding behavioral-finance models' presumption that participants' optimism (pessimism) leads them to forego profit opportunities.*

(ii) *KUH representations are compatible with the diversity of market sentiment's influence on individual participant's forecasts.*

(iii) *These representations recognize that whether the market is in an optimistic, neutral, or pessimistic state, and how this state influences participants' forecasts, changes at times and in ways that cannot be represented with a stochastic process.*

The Empirical Consequences of Hypothesis 1 According to the no-arbitrage condition in (38), and given the representations in (58)-(59) , (62)-(63), optimistic (pessimistic) participants bid the stock price to lie within the upper (lower) subintervals of the no-arbitrage interval, $\mathcal{I}_t^p$ in (41):

$$p_t^{opt} \in \mathcal{I}_t^p(s_t = 1) = \gamma \left(I_{t:t+1}^{\varphi}(s_t = 1) + I_t^{\phi}(s_t = 1) \right) x_t, \tag{66}$$

$$p_t^{pes} \in \mathcal{I}_t^p(s_t = -1) = \gamma \left(I_{t:t+1}^{\varphi}(s_t = -1) + I_t^{\phi}(s_t = -1) \right) x_t, \tag{67}$$

where p_t^{opt} and p_t^{pes} denote the values of prices set by the market when its participants are optimistic or pessimistic.

The subintervals, within which p_t^{opt} and p_t^{opt} in (66) and (67) lie, depend on x_t, the model's KU parameters $\left(\mu_-, \mu_+, \rho_\mu, b_-, b_+, \rho_b\right)$, and the sentiment effect, η. Thus, contingent on whether the market is optimistic or pessimistic, we can confront the model's predictions about the influence of sentiment on participants' forecasts by assessing whether the time-series observations on p_t actually lie within the subintervals (66) and (67). In Section 9 and Appendix B.4, we illustrate how this can be done using an econometric calibration methodology and the proxy for market sentiment extracted from narrative market reports.

Remark 10 *Representations in (66) and (67) highlight the essential role of Muth's hypothesis in building intertemporal models under Knightian uncertainty. Imposing consistency within a KUH model enables an economist to represent and test the influence of non-fundamental factors (market sentiment) on aggregate outcomes (stock prices).*

8.4.2 Market Sentiment in Participants' Forecast Revisions

As we demonstrated in Section 6, although a KUH model generates quantitative predictions about the interval within which the values of p_t lie at a point in time, conditional on x_t, the model does not generate quantitative predictions about how p_t and x_t co-move over time. Moreover, a KUH model's qualitative predictions about the co-movements in time-series data are contingent on whether and, if so, how the model specifies change in its representations of participants' forecasts.[32] For example, as is apparent from (52), leaving changes in the coefficients, $\Delta\varphi_t$ and $\Delta\phi_t$, in the representations of $\mathcal{F}_t\left(d_{t+1}\right)$ and $\mathcal{F}_t\left(p_{t+1}\right)$ unconstrained renders even the model's qualitative predictions ambiguous, in the sense that the model is compatible with both positive and negative co-movement between p_{t+k} and x_{t+k}, at any time horizon k.[33]

Remark 11 *The ambiguity of a KUH model's predictions is just another way of stating that, under Knightian uncertainty, the coefficients of the model-consistent*

[32] We refer to a macroeconomic model's predictions as contingent if they depend on some variables (events), the effects of which cannot be specified *ex ante* with a stochastic process. Recognizing that an economist faces Knightian uncertainty means that he cannot specify completely how future events will affect the process driving outcomes. Thus, under Knightian uncertainty, predictions of co-movements in time-series data are necessarily contingent. An exploration of this point and related issues is beyond the scope of this paper.

[33] This is in contrast to REH models, which generate unambiguous quantitative and qualitative time-t predictions of the co-movement between p_{t+k} and x_{t+k}, for each time horizon k.

representations of participants' forecasts are partly autonomous: they are not completely determined in terms of the model's KU parameters, $\left(\mu_-, \mu_+, \rho_\mu, b_-, b_+, \rho_b\right)$, its coefficients at time t, μ_t and b_t, and the moments of its stochastic innovations.

The autonomy of a KUH model's representation of participants' forecasts thus reveals one of the key implications of recognizing an economist's Knightian uncertainty in an intertemporal model. For a consistent model to generate even qualitative predictions of the co-movements in time-series data, an economist must appeal to a non-fundamental factor and formalize its effect with constraints on the change in the parameters of his model's representation of participants' forecasts.

Given φ_{t-1} and ϕ_{t-1}, constraining $\Delta\varphi_t = \varphi_t - \varphi_{t-1}$ and $\Delta\phi_t = \phi_t - \phi_{t-1}$ involves constraining φ_t and ϕ_t to lie within the subintervals $I^{\varphi}_{t:t+1}$ and I^{ϕ}_t specified in (31) and (46). Here, we consider a particularly simple example of such constraints: optimistic (pessimistic) participants' revisions of forecasts, in terms of earnings, are represented by constraining $\Delta\varphi_t > 0$ and $\Delta\phi_t > 0$ ($\Delta\varphi_t < 0$ and $\Delta\phi_t < 0$).

8.4.3 Representations of Participants' Forecast Revisions

We next consider Hypothesis 2 given by:

Hypothesis 2

(i) If the market is optimistic at time t, that is, if $s_t = 1$, and $\varphi_{t-1} < U^{\varphi}_{t:t+1}$ and $\phi_{t-1} < U^{\phi}_t$, then

$$\varphi^{opt}_t \in [\max(\varphi_{t-1}, L^{\varphi}_{t:t+1}), U^{\varphi}_{t:t+1}], \tag{68}$$

$$\phi^{opt}_t \in [\max(\phi_{t-1}, L^{\phi}_t), U^{\phi}_t]. \tag{69}$$

(ii) If the market is pessimistic at time t, that is, if $s_t = -1$, and $\varphi_{t-1} > L^{\varphi}_{t:t+1}$ and $\phi_{t-1} > L^{\phi}_t$, then

$$\varphi^{pes}_t \in [L^{\varphi}_{t:t+1}, \min(\varphi_{t-1}, U^{\varphi}_{t:t+1})], \tag{70}$$

$$\phi^{pes}_t \in [L^{\phi}_t, \min(\phi_{t-1}, U^{\phi}_t)]. \tag{71}$$

We note that $(\varphi^{opt}_t, \varphi^{pes}_t, \phi^{opt}_t, \phi^{pes}_t)$ lie within their respective model-consistent intervals, if the constraints $\varphi_{t-1} < U^{\varphi}_{t:t+1}$ and $\phi_{t-1} < U^{\phi}_t$ ($\varphi_{t-1} > L^{\varphi}_{t:t+1}$ and

$\phi_{t-1} > L_t^{\phi}$) hold. If these constraints are not satisfied, it is impossible that $\phi_t \in I_t^{\phi}$ such that $\Delta\phi_t > 0$ when $s_t = 1$. Likewise for φ_t. The following lemma establishes sufficient conditions for the constraints.

Lemma 5 *If $\mu_t > \mu_{t-1}$, in (1) and $b_t > b_{t-1}$ in (14) the constraints $\varphi_{t-1} < U_{t:t+1}^{\varphi}$ and $\phi_{t-1} < U_t^{\phi}$ in Hypothesis 2 are satisfied. Analogously if $\mu_t < \mu_{t-1}$ and $b_t < b_{t-1}$, the constraints $\varphi_{t-1} > L_{t:t+1}^{\varphi}$ and $\phi_{t-1} > L_t^{\phi}$ are satistfied.*

Remark 12 *Lemma 5 reveals the theoretical importance of behavioral finance's empirical findings that non-fundamental factors exert an autonomous, significant influence on how market participants revise their forecasts. The relevance of factors such as market sentiment may enable economists to build models that generate empirically verifiable predictions of co-movements in time-series data.*[34]

The Empirical Consequences of Hypothesis 2 Applying Hypothesis 2 to constrain $\Delta\varphi_t$ and $\Delta\phi_t$ in the expression for change in the stock price, $\Delta p_t = \gamma\left(\varphi_t + \phi_t\right)\Delta x_t + \gamma\left(\Delta\varphi_t + \Delta\phi_t\right)x_{t-1}$, illustrates one such prediction:

(i) If $s_t = 1$ and $\Delta x_t > 0$, then (68) and (69) imply that $\Delta p_t > 0$.

(ii) If $s_t = -1$ and $\Delta x_t < 0$, then (70) and (71) imply that $\Delta p_t < 0$.

9 Confronting a KUH Model's Predictions with Time-Series Data: An Illustration of the Econometric Methodology with the S&P Stock-Price Index

Recognizing that economists face Knightian uncertainty about how outcomes unfold over time, poses considerable challenges for assessing the empirical relevance of macroeconomic and finance models' predictions. Here, we sketch how the existing econometric methods, particularly econometrically-based calibration, as advocated by Hansen and Heckman (1996), can be adapted to meet this challenge.

[34] The conditions under which the constraints in (68)-(71) are compatible with model consistency at $t-1$ and t , depend on the change in the model's coefficients and its KU parameters.

We focus on our prototype's quantitative predictions that stock prices lie within the no-arbitrage intervals that depend on earnings. The details of our calibration methodology, econometric specifications, graphs and tables of the results are presented in the Appendix B.

The core premise of KUH is that any fixed stochastic model eventually ceases to approximate time-series data adequately, owing to unforeseeable structural change in the process driving aggregate outcomes. While such change cannot be represented *ex ante* with a stochastic process, it can be approximated *ex post* on the basis of historical time-series data. As new data accrue, the econometric model must be re-estimated, new potential structural changes must be identified, and the adequacy of the re-estimated model must be assessed.

In our econometrically-based calibration approach, we build on the generalized autoregressive score (GAS) approach.[35] We estimate approximations of earnings and dividend processes, in (1) and (14), for the sample of stock prices and earnings of the companies included in the S&P 500 Index, spanning the period from 1960(4) to 2017(3). These approximations allow for both time-varying coefficients and structural breaks. We rely on standard misspecification tests to assess the adequacy of the econometric model as an approximation of the data. This enables us to suggest estimates of the sequences $\{\mu_t, b_t\}_{t=1,2,\ldots,T}$.

Given that the estimated econometric model is an adequate approximation of the past data, we can assess the quantitative predictions of the KUH model. All these predictions depend on the Knightian uncertainty parameters $\left(\mu_-, \mu_+, \rho_\mu, b_-, b_+, \rho_b\right)$. To compute the empirical counterpart of the stock-price interval $\mathcal{I}_t^p$ in (41), we use the estimated sequences $\{\hat{\mu}_t, \hat{b}_t\}_{t=1,2,\ldots,T}$, and we choose values for the Knightian uncertainty parameters, the discount factor γ, and v. This leads us to compute how often the observed stock prices take values within these empirical intervals. As described in Appendix B.3 we find that the observed stock price p_t lies in the empirical stock price intervals in 96 percent of the observations.

To assess the KUH model's predictions contingent on market sentiment, we rely on a numerical proxy for sentiment. This enables us to assess the adequacy of the empirical implications of Hypothesis 1, in Section 8.4.1, by computing, contingent

[35] Creal *et al*. (2012) gives an overview of the GAS approach and its applications.

on whether the market is optimistic ($s_t = 1$) or pessimistic ($s_t = -1$), how often the observed stock prices are within the upper (lower) subintervals, in (66) or (67) The results are summarized in Appendix B.4. We find that the observed stock price lies in the respective intervals in 74 of the 76 observations where $s_t = 1$ or $s_t = -1$ for $\eta = 0.2$. Finally, we assess the adequacy of the empirical implications of Hypothesis 2, in Section 8.4.3 by computing how often, contingent on whether $s_t = 1$ ($s_t = -1$) and $\Delta x_t > 0$ ($\Delta x_t < 0$), the observed stock prices co-move positively with earnings. We find that the stock price increases in 21, or 75 percent, of the 28 observations where earnings increase and the market is optimistic. The stock price decreases in nine, or 60 percent, of the 15 observations where earnings decrease and the market is pessimistic.

Rigorous assessment of the consequences of Hypothesis 1 and 2 as well as other representations of the influence of non-fundamental factors, such market sentiment, requires the development of a methodology for testing models that formalize both measurable and Knightian uncertainty about the process driving aggregate outcomes. Thus, although the results we present are broadly supportive of a KUH prototype's predictions, we view them as strictly preliminary.

An assessment of the empirical relevance of the KUH present-value model of stock prices also requires developing extensions of our prototype model, which would allow for the time-varying discount factor and generalize the model's other simplifying specifications. However, illustrating how our prototype's predictions can be confronted with time-series data has enabled us to highlight some of the essential features of the econometric methodology needed to test models that recognize that an econometrician faces Knightian uncertainty, which, by definition, cannot be characterized with a stochastic process.

10 Concluding Remarks

For Knight, recognizing unforeseeable change and the true uncertainty that such change engenders is the key to understanding profit-seeking activity in real-world markets. As he put it:

"if all changes [...] could be foreseen for an indefinite period in advance of their occurrence [...] profit or loss would not arise" (Knight 1921, p. 198).

But if Knight is correct in arguing that an inherent feature of profit seeking is that market participants are alert to unforeseeable change and revise their decision-making accordingly, we would expect that the process driving aggregate outcomes that result from participants' demand and supply decisions undergoes unforeseeable change as well.

To be sure, econometric analysis cannot decisively reject REH and behavioral-finance models' core premise that an economist can represent change in an economy's structure over an indefinite future with probabilistic rules. After all, "unforeseeable change" refers to the possibility of representing future change, whereas econometric analysis, *ipso facto*, can ascertain only whether particular models missed structural changes that an econometrician had not specified in the past.

There is ample evidence that the process driving outcomes, especially in asset markets, undergoes quantitative structural change. The key question regarding the empirical relevance of Knightian uncertainty is how to ascertain whether this structural change is, at least in part, unforeseeable. The findings of a number of econometric studies point to the key reason why this is the case: structural change in models of outcomes, especially in financial markets, seems to occur contemporaneously with historical events that are not exact repetitions of similar events in the past. These events give rise to change in the economy's structure that could not have been represented *ex ante* with probabilistic rules.[36]

Kaminsky's (1993) largely overlooked study of currency markets shows that historical events may trigger change in the parameters of the probabilistic rule, which is often used to represent change in the process driving outcomes. She finds that the Markov model's transition probabilities are not only time-varying, but that they also depend on who is Fed chair and the credibility of the incumbent's policies. Moreover, Kaminsky shows that the predictions of Engel and Hamilton's (1990) Markov model, which ignores such change, are inconsistent with the actual turning points in currency movements. Frydman and Goldberg's (2007) analysis of structural change in major currency markets lends support to Kaminsky's conclusion that

[36]For evidence that structural change in models of stock returns is related to historical events that are to some extent novel, see Pettenuzzo and Timmermann (2011) and Ang and Timmermann (2012). Frydman *et al*. (2015) provide evidence that 20% of events that triggered movements in US stock prices between 1993 and 2009 were, at least in part, non-repetitive.

historical events are among the major triggers of inflection points.

In a series of papers and books, David Hendry has traced macroeconomic models' empirical difficulties to their structural instability. He has demonstrated not only that macroeconomic models experience structural breaks, but also that these breaks are often triggered by historical events. The novel mathematical framework that underpins KUH enables economists to specify models of outcomes that undergo such inherently unforeseeable structural change.[37]

Moreover, relying on this framework and Muth's hypothesis, KUH offers a coherent approach to building intertemporal macroeconomics and finance models that recognize that economists as well as market participants face Knightian uncertainty about the process driving aggregate outcomes. KUH thus opens a way to construct macroeconomics and finance models premised on market participants' rationality that accord a role to both fundamental and psychological (and other non-fundamental) considerations in driving aggregate outcomes. We have provided examples of how a KUH model can represent the autonomous influence of market sentiment on participants' forecasts, leaving a more complete presentation of "behavioral finance under Knightian uncertainty" to a follow-up paper. Much work remains to be done to develop KUH models that specify key features of processes driving outcomes in specific contexts or markets.

We have also suggested how the existing econometric methods – involving calibration, estimation of models with time-varying parameters, and reliance on information extracted from narrative market reports – can be adapted to confront KUH models with time-series data. The development of the statistical methodology needed to test models of outcomes characterized by Knightian uncertainty is another area that we plan to explore in future research.

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Appendix (For online publication)

A Proofs of Lemmas and Theorems

Proof of Lemmas 1 and 3. Proof of (8) and (9): We give the proof of

$$\mu_{t+k} \geq L^{\mu}_{t:t+k} = \mu_- + \rho^k_\mu \left(\mu_t - \mu_-\right),$$

because the proofs for $U^{\mu}_{t:t+k}$, $L^{b}_{t:t+k}$ and $U^{b}_{t:t+k}$ are similar. For $k = 1$, this follows from Assumption 1 and the general result by induction. We give the proof for $k = 2$. We use the result for $k = 1$ and t replaced by $t + 1$, and find

$$\begin{aligned}\mu_{t+2} &\geq \mu_- + \rho_\mu \left(\mu_{t+1} - \mu_-\right) \geq \mu_- + \rho_\mu \left(\mu_- + \rho_\mu \left(\mu_t - \mu_-\right) - \mu_-\right) \\ &= \mu_- + \rho^2_\mu \left(\mu_t - \mu_-\right) = L^{\mu}_{t:t+2},\end{aligned}$$

which completes the proof. ■

Proof of Lemma 4. Proof of (24): From

$$d_{t+1} = b_{t+1} x_{t+1} + \varepsilon_{d,t+1} = b_{t+1} x_t \exp\left(\varepsilon_{x,t+1} + \mu_{t+1}\right) + \varepsilon_{d,t+1},$$

we apply the bounds for b_{t+1} and μ_{t+1} and find

$$d_{t+1} \geq L^{b}_{t:t+1} x_t \exp\left(\varepsilon_{x,t+1} + L^{\mu}_{t:t+1}\right) + \varepsilon_{d,t+1} = L^{b}_{t:t+1},$$

which proves the result for $k = 1$. The general proof is by induction. We give the proof for $k = 2$. We apply the result for $k = 1$ and t replaced by $t + 1$, and find using,

$$L^{\mu}_{t+1:t+2} = \mu_- + \rho_\mu \left(\mu_{t+1} - \mu_-\right) \geq \mu_- + \rho^2_\mu \left(\mu_t - \mu_-\right) = L^{\mu}_{t:t+2},$$

that

$$
\begin{aligned}
d_{t+2} &\geq L^b_{t+1:t+2} x_{t+1} \exp\left(\varepsilon_{x,t+2} + L^{\mu}_{t+1:t+2}\right) + \varepsilon_{d,t+2} \\
&= L^b_{t+1:t+2} x_t \exp\left(\varepsilon_{x,t+1} + L^{\mu}_{t:t+1}\right) \exp\left(\varepsilon_{t+2} + L^{\mu}_{t+1:t+2}\right) + \varepsilon_{d,t+2} \\
&\geq x_t \exp(\sum_{j=1}^{2} \varepsilon_{x,t+j}) L^b_{t:t+2} \exp(\sum_{j=1}^{2} L^{\mu}_{t:t+j}),
\end{aligned}
$$

which completes the proof. ■

Proof of Theorem 2. That (30) and (31) apply follows from the definition of $KE_t\left(d_{t+k}\right)$ in (26) and the expressions for the end-points of $I^d_{t:t+k}$ in (28) and (29).

The proof of (32) follows by induction. Consider the case of $k = 2$,

$$KE_t\left(KE_{t+1}\left(d_{t+2}\right)\right).$$

Using (31) for $t + 2$, $KE_{t+1}\left(d_{t+2}\right)$ is the interval given by

$$E_{t+1}\left(I^d_{t+1:t+2}\right) = \left[E_{t+1}\left(L^d_{t+1:t+2}\right), E_{t+1}\left(U^d_{t+1:t+2}\right)\right],$$

where, again by definition,

$$
\begin{aligned}
E_{t+1}\left(L^d_{t+1:t+2}\right) &= x_{t+1} v L^b_{t+1:t+2} \exp\left(L^{\mu}_{t+1:t+2}\right) \\
&= x_t \exp\left(\mu_{t+1} + \varepsilon_{x,t+1}\right) v L^b_{t+1:t+2} \exp\left(L^{\mu}_{t+1:t+2}\right),
\end{aligned}
$$

and likewise for $E_{t+1}\left(U^d_{t+1:t+2}\right)$. Thus, the end-points of the interval $KE_{t+1}\left(d_{t+2}\right)$ contain μ_{t+1} and b_{t+1} which have not been introduced at time t. We bound the interval $KE_{t+1}\left(d_{t+2}\right)$, and find that $KE_{t+1}\left(d_{t+2}\right)$ is contained in an interval, with lower end-point,

$$x_t \exp\left(L^{\mu}_{t:t+1} + \varepsilon_{x,t+1}\right) v L^b_{t:t+2} \exp\left(L^{\mu}_{t:t+2}\right).$$

Similarly for the upper end-point. Collecting terms, we find,

$$\begin{aligned}&KE_t\left(KE_{t+1}\left(d_{t+2}\right)\right)=KE_t\left(E(I^d_{t+1:t+2}\right)\\=\,&x_t v^2\left[b_{L,t:t+2}\exp\left(\mu_{L,t:t+1}+\mu_{L,t:t+2}\right),b_{U,t:t+2}\exp\left(\mu_{U,t:t+1}+\mu_{U,t:t+2}\right)\right].\end{aligned}$$

The identity,

$$KE_t\left(KE_{t+1}\left(d_{t+2}\right)\right)=KE_t\left(d_{t+2}\right),$$

can be seen by using the monotonicity properties in (23),

$$d_{t+2}\in I^d_{t+1:t+2}\subseteq I^d_{t:t+2}=\left[L^d_{t:t+2},U^d_{t:t+2}\right],$$

with

$$\begin{aligned}L^d_{t:t+2}&=x_t\exp\left(\varepsilon_{x,t+1}+\varepsilon_{x,t+2}\right)L^b_{t:t+2}\exp\left(L^\mu_{t:t+1}+L^\mu_{t:t+2}\right)+\varepsilon_{d,t+2},\\U^d_{t:t+2}&=x_t\exp\left(\varepsilon_{x,t+1}+\varepsilon_{x,t+2}\right)U^b_{t:t+2}\exp\left(U^\mu_{t:t+1}+U^\mu_{t:t+2}\right)+\varepsilon_{d,t+2},\end{aligned}$$

which completes the proof. ■

Proof Theorem 3. Note that by definition, $\mathcal{I}^p_{t+1}=\sum_{i=1}^{\infty}\gamma^i KE_t\left(d_{t+1+i}|X_{t+1}\right)$ where $KE_t\left(d_{t+1+i}\right)$ has lower and upper end-points given by,

$$v^i x_{t+1}L^b_{t+1:t+1+i}\exp(\sum_{j=1}^{i}L^\mu_{t+1:t+1+j}),\quad\text{and}\quad v^i x_{t+1}U^b_{t+1:t+1+i}\exp(\sum_{j=1}^{i}U^\mu_{t+1:t+1+j}),$$

which are functions of μ_{t+1}, b_{t+1} and $\varepsilon_{x,t+1}$. Using Lemmas 1 and 3 the lower end-point can be bounded by,

$$v^i x_{t+1}L^b_{t+1:t+1+i}\exp(\sum_{j=1}^{i}L^\mu_{t+1:t+1+j})\geq v^i x_t\exp\left(\varepsilon_{x,t+1}\right)L^b_{t:t+1+i}\exp(\sum_{j=0}^{i}L^\mu_{t:t+1+j}).$$

Collecting terms, it follows that $\mathcal{I}^p_{t+1}$ has the lower bound given by,

$$\sum_{i=1}^{\infty}\gamma^i v^i x_t\exp\left(\varepsilon_{x,t+1}\right)L^b_{t:t+1+i}\exp(\sum_{j=1}^{i+1}L^\mu_{t:t+j}).$$

Hence the lower end-point of $KE_t\left(\mathcal{I}^p_{t+1}\right)$ is given by,

$$\sum_{i=1}^{\infty}\gamma^i v^{i+1}x_t L^b_{t:t+1+i}\exp(\sum_{j=1}^{i+1}L^\mu_{t:t+j}).$$

Next, recall that $KE_t\left(d_{t+1}\right)$ has the lower end-point,

$$vx_t L^b_{t:t+1}\exp\left(L^\mu_{t:t+1}\right),$$

such that the lower end-point of the right hand side of (40) therefore is given by,

$$\begin{aligned}&\gamma(vx_t L^b_{t:t+1}\exp\left(L^\mu_{t:t+1}\right)+\sum_{i=1}^{\infty}\gamma^i v^{i+1}x_t L^b_{t:t+1+i}\exp(\sum_{j=1}^{i+1}L^\mu_{t:t+j}))\\ =&\sum_{i=1}^{\infty}\gamma^i v^i x_t L^b_{t:t+i}\exp(\sum_{j=1}^{i}L^\mu_{t:t+j}),\end{aligned}$$

which is the lower end-point of $\mathcal{I}^p_t$ as desired. Similarly for the upper end-point which proves the claimed result. ■

Proof Lemma 5. We prove the result for the upper bounds. From (8), (22), (31) and (43) we find the expressions

$$\begin{aligned}U^\mu_{t:t+k}&=\mu_+ + \rho^k_\mu(\mu_t-\mu_+),\\ U^b_{t:t+k}&=b_+ + \rho^k_b(b_t-b_+),\\ U^\varphi_{t:t+1}&=vU^b_{t:t+1}\exp(U^\mu_{t:t+1}),\\ U^\phi_t&=\sum_{k=1}^{\infty}\gamma^i v^k U^b_{t:t+k}\exp(\sum_{j=1}^{k}U^\mu_{t:t+j}).\end{aligned}$$

It is seen that $U^\mu_{t:t+k}$ depends linearly on μ_t with a positive coefficient, ρ^k_μ, so that $U^\mu_{t:t+k}$ is increasing in μ_t, such that if $\mu_{t-1}<\mu_t$,

$$U^\mu_{t-1:t-1+k}=\mu_+ + \rho^k_\mu(\mu_{t-1}-\mu_+)<\mu_+ + \rho^k_\mu(\mu_t-\mu_+)=U^\mu_{t:t+k}.$$

A similar expression shows that $U^b_{t:t+k}$ is increasing in b_t. It follows that $U^\varphi_{t:t+1}$ and

U_t^{ϕ} are increasing functions of both μ_t and b_t. Thus, for $\mu_{t-1} < \mu_t$ and $b_{t-1} < b_t$, it follows that $U_{t-1:t}^{\varphi} < U_{t:t+1}^{\varphi}$ and $U_{t-1}^{\phi} < U_t^{\phi}$.

A consequence of $U_{t-1:t}^{\varphi} < U_{t:t+1}^{\varphi}$ is that

$$\varphi_{t-1} \leq U_{t-1:t}^{\varphi} < U_{t:t+1}^{\varphi},$$

which completes the proof. ■

B Econometric Methodology, Data and Results

We confront the KUH prototype's predictions with time-series data in the context of approximations of earnings and dividend processes, in (1) and (14), for the sample of stock prices and earnings of the companies included in the S&P 500 Index, spanning the period from 1960(4) to 2017(3). The data are described in Appendix B.5.

B.1 Econometric Models

In our econometric approach, we specify time-varying coefficients equivalent to μ_t and b_t. To this end, we build on the generalized autoregressive score (GAS) approach by including structural breaks. We rely on standard misspecification tests to assess the adequacy of the econometric model as an approximation of the data. This allows us to suggest estimates of the sequences $\{\mu_t, b_t\}_{t=1,2,...,T}$.

There are, of course, potentially many other econometric models that might approximate the historical time-series data. In principle, one could estimate several econometric models; as long as they provide adequate approximations of the historical data, the KUH model's predictions should hold.

Specifically, we first consider modeling log-changes in earnings as:

$$\Delta \log x_t = \mu_t + \gamma_x' F_{x,t} + \varepsilon_{x,t}, \tag{B.1}$$

$$\mu_t = \tilde{\mu}_t + \delta_x' S_{x,t}, \tag{B.2}$$

$$\tilde{\mu}_t = \omega_\mu + \alpha_{\mu 1}\tilde{\mu}_{t-1} + \alpha_{\mu 2}\tilde{\mu}_{t-2} + \beta_\mu \varepsilon_{x,t-1}, \tag{B.3}$$

for $t = 1, 2, \ldots, T$, where $\varepsilon_{x,t} \sim i.i.d.N\left(0, \sigma_x^2\right)$ and the initial value are set to

$\mu_0 = \Delta \log x_0$ and $\mu_{-1} = \Delta \log x_{-1}$. The vector $F_{x,t}$ includes a set of six dummy variables corresponding to the observations of extreme changes in earnings from 2008 to 2010 as evident from Panel (c) in Figure 4 in Appendix B.5. The vector $S_{x,t}$ consists of 12 subsample dummies that take the value 1 during specific subperiods, and zero otherwise. In the estimations, we treat the variables $F_{x,t}$ and $S_{x,t}$ as fixed. Prior to estimation of the model in (B.1)-(B.3), the subsample dummy variables in $S_{x,t}$ have been selected using the Autometrics algorithm in OxMetrics.[38]

We model dividends as:

$$d_t = b_t x_t + \gamma_d' F_{d,t} + \varepsilon_{d,t}, \tag{B.4}$$

$$b_t = \tilde{b}_t + \delta_d' S_{d,t}, \tag{B.5}$$

$$\tilde{b}_t = \omega_b + \alpha_{b1} \tilde{b}_{t-1} + \alpha_{b2} \tilde{b}_{t-2} + \beta_b \frac{\varepsilon_{d,t-1}}{x_{t-1}}, \tag{B.6}$$

for $t = 1, 2, \ldots, T$, where $\varepsilon_{d,t} \sim i.i.d.N\left(0, \sigma_d^2\right)$ and the initial values are set to $b_0 = d_0/x_0$ and $b_{-1} = d_{-1}/x_{-1}$. The vector $F_{d,t}$ includes a set of four dummy variables for the observations during the financial crisis, while $S_{d,t}$ is a vector of subsample dummies that take the value 1 during specific subperiods, and zero otherwise. Prior to estimation of the model in (B.4)-(B.6), the subsample dummy variables in $S_{d,t}$ have been selected using the Autometrics algorithm in OxMetrics, see Doornik (2009). Details and a full description of $F_{d,t}$ and $S_{d,t}$ are given in Appendix B.6.

The models in (B.1)-(B.3) and (B.4)-(B.6) specify the time-varying coefficients μ_t and b_t as observation-driven autoregressive processes combined with structural breaks in the levels due to the inclusion of the subsample dummies $S_{x,t}$ and $S_{d,t}$.

B.2 Empirical Results

We estimate the models in (B.1)-(B.3) and (B.4)-(B.6) by Gaussian maximum likelihood using time-series data for the real S&P500 dividends and earnings, which generates an effective sample of $T = 228$ observations covering the period from 1960(4) to 2017(3). Plots of the time-series data are shown in Figure 4 in Appendix B.5. To assess the adequacy of the econometric model as an approximation of the

[38] See Doornik (2009). Details, including a full description of $F_{x,t}$ and $S_{x,t}$, are provided in Appendix B.6.

historical data, we rely on standard misspecification tests.

The estimation results and misspecification tests are shown in Table 1. For the model of the log-changes in earnings, the tests for no autocorrelation of order 1 and order 1-4 are not rejected with p-values of 0.30 and 0.15. Moreover, the test for no ARCH of order 1-4 is not rejected with a p-value of 0.22, and normality of the estimated residuals is not rejected with a p-value of 0.48. As the misspecification tests are not rejected, we conclude that the estimated model is an adequate approximation of the log-change in earnings over the sample period considered. Importantly, restricting $\delta_x = 0$ – that is, assuming that there are no structural breaks in the time-varying coefficient μ_t – renders the model inadequate as an approximation of the earnings process.

For the model of dividends, the tests for no autocorrelation of order 1 and order 1-4 are not rejected with p-values of 0.97 and 0.75, respectively.[39] Importantly, restricting $\delta_d = 0$ – that is, assuming that there no structural breaks in the time-varying coefficient b_t – renders the model inadequate as an approximation of the dividends process.

B.3 Empirical Stock-Price Intervals

Given the estimated sequences $\{\hat{\mu}_t, \hat{b}\}_{t=1,2,\ldots,T}$ and values for the parameters ρ_μ, μ_-, μ_+, ρ_b, b_-, b_+, γ, v and σ_x^2, we can compute the empirical counterparts of the intervals $I^\mu_{t:t+1}$ in (7), $I^b_{t:t+1}$ in (20), and the stock-price interval $\mathcal{I}^p_t$ in (41).

We first use the estimates in Table 1 to set the parameters ρ_μ, ρ_b, and v. We set the parameter ρ_μ to the modulus of the largest inverse root of the characteristic polynomial for $\tilde{\mu}_t$, given by $\alpha_\mu(z) = 1 - \alpha_{1\mu} z - \alpha_{2\mu} z^2$. Equivalently, we set the parameter ρ_b to the modulus of the largest inverse root of the characteristic polynomial for $\tilde{b}_t$, given by $\alpha_b(z) = 1 - \alpha_{1b} z - \alpha_{2b} z^2$. That gives the parameter values $\rho_\mu = 0.743$ and $\rho_b = 0.738$. Moreover, we use the estimate $\sigma_x = 0.028$ to set the parameter value $v = E\exp(\varepsilon_{x,t}) = \exp(-\sigma_x^2/2) = 0.9996$ and we set the discount factor to $\gamma = 0.94$.

[39]However, no ARCH and normality of the estimated residuals are both rejected with p-values of 0.000. This is caused by a few potential outliers and a tendency for the variance σ_d^2 to increase over the sample period, which have not been accounted for in the model. We leave an investigation of these specification problems for future development of a model of stock prices.

Model for earnings in (1)–(3)			Model for dividends in (4)–(6)		
Coefficient	Estimate	Std.Error	Coefficient	Estimate	Std.Error
$\gamma_x^{(t=2008(4))}$	-1.071	0.030	$\gamma_d^{(t=2008(4))}$	1.222	0.058
$\gamma_x^{(t=2009(1))}$	-0.795	0.033	$\gamma_d^{(t=2009(1))}$	3.248	0.065
$\gamma_x^{(t=2009(2))}$	0.052	0.035	$\gamma_d^{(t=2009(2))}$	2.731	0.048
$\gamma_x^{(t=2009(3))}$	0.481	0.035	$\gamma_d^{(t=2009(3))}$	1.326	0.054
$\gamma_x^{(t=2009(4))}$	1.381	0.036			
$\gamma_x^{(2010(1)\leq t\leq 2010(2))}$	0.098	0.028			
$\delta_x^{(1987(3)\leq t\leq 1988(2))}$	0.110	0.023	$\delta_d^{(1972(3)\leq t\leq 1981(2))}$	-0.003	0.016
$\delta_x^{(1988(3)\leq t\leq 1991(4))}$	-0.033	0.014	$\delta_d^{(1981(3)\leq t\leq 1990(4))}$	-0.011	0.022
$\delta_x^{(1992(1)\leq t\leq 2000(3))}$	0.028	0.010	$\delta_d^{(1991(1)\leq t\leq 1994(1))}$	-0.024	0.026
$\delta_x^{(2000(4)\leq t\leq 2001(1))}$	-0.073	0.027	$\delta_d^{(1994(2)\leq t\leq 1999(3))}$	-0.027	0.029
$\delta_x^{(2001(2)\leq t\leq 2001(3))}$	-0.199	0.031	$\delta_d^{(1999(4)\leq t\leq 2001(2))}$	-0.034	0.030
$\delta_x^{(2001(4)\leq t\leq 2002(1))}$	-0.053	0.030	$\delta_d^{(2001(3)\leq t\leq 2003(1))}$	0.023	0.033
$\delta_x^{(2002(2)\leq t\leq 2002(3))}$	0.075	0.032	$\delta_d^{(2003(2)\leq t\leq 2003(3))}$	-0.066	0.034
$\delta_x^{(2002(4)\leq t\leq 2002(4))}$	-0.067	0.030	$\delta_d^{(2003(4)\leq t\leq 2007(3))}$	-0.108	0.038
$\delta_x^{(2003(1)\leq t\leq 2003(4))}$	0.156	0.025	$\delta_d^{(2007(4)\leq t\leq 2008(1))}$	-0.058	0.038
$\delta_x^{(2004(1)\leq t\leq 2007(2))}$	0.013	0.015	$\delta_d^{(2008(2)\leq t\leq 2008(3))}$	-0.012	0.042
$\delta_x^{(2007(3)\leq t\leq 2008(3))}$	-0.099	0.023	$\delta_d^{(2009(4)\leq t\leq 2013(1))}$	-0.062	0.048
$\delta_x^{(2010(3)\leq t\leq 2017(3))}$	0.016	0.011	$\delta_d^{(2013(2)\leq t\leq 2015(1))}$	-0.058	0.048
			$\delta_d^{(2015(2)\leq t\leq 2017(3))}$	-0.063	0.048
ω_μ	-0.001	0.003	ω_b	0.035	0.010
$\alpha_{1\mu}$	0.953	0.069	α_{1b}	1.474	0.046
$\alpha_{2\mu}$	-0.551	0.079	α_{2b}	-0.545	0.045
β_μ	0.559	0.066	β_b	1.490	0.064
σ_x	0.028		σ_d	0.798	
Log-likelihood	488.359		Log-likelihood	-217.965	
Ljung-Box 1-1	$\chi^2(1) = 1.08$ [0.30]		Ljung-Box 1-1	$\chi^2(1) = 0.00$ [0.97]	
Ljung-Box 1-4	$\chi^2(4) = 6.72$ [0.15]		Ljung-Box 1-4	$\chi^2(4) = 1.94$ [0.75]	
No ARCH 1-4	$F(4, 219) = 1.44$ [0.22]		No ARCH 1-4	$F(4, 219) = 5.13$ [0.00]	
Normality	$\chi^2(2) = 1.47$ [0.48]		Normality	$\chi^2(2) = 52.28$ [0.00]	

Table 1: The table shows the estimates of the model for the log-change in earnings in (B.1)-(B.3) and the model for dividends in (B.4)-(B.6). Both models are estimated for an effective sample of $T = 228$ observations covering the sample from 1960(4) to 2017(3). P-values in square brackets for the misspecification tests.

We next calibrate the parameters (μ_-, μ_+, b_-, b_+) such that the empirical stock-price intervals $\widehat{\mathcal{I}}_t^p$ match the range of historical stock prices, and such that a sufficiently high percentage of $(\hat{\mu}_{t+1}, \hat{b}_{t+1})$, given the values of $(\hat{\mu}_t, \hat{b}_t)$, lie within the computed empirical intervals for $I^{\mu}_{t:t+1}$ and $I^{b}_{t:t+1}$. Moreover, we set $\mu_+ = 0.047$ to ensure that the transversality condition of the theoretical model is satisfied. Not taking the uncertainty of $\hat{\mu}_t$ into account, however, we note that almost ten percent of the estimates $\hat{\mu}_t$ lie above this value. The parameter values are shown in Table 2.

We compute the empirical intervals $\hat{I}^{\mu}_{t:t+1}$ and $\hat{I}^{b}_{t:t+1}$ given the parameter values in Table 2 and the estimates $\hat{\mu}_t$ and $\hat{b}_t$. Panel (a) in Figure 1 shows the estimates of $\hat{\mu}_{t+1}$ (red line) and the computed empirical intervals $\hat{I}^{\mu}_{t:t+1}$ (vertical grey lines). We find that $\hat{\mu}_{t+1}$ lies within these intervals in 139, or 61 percent, of the 228 observations. Panel (b) in Figure 1 shows the estimates of $\hat{b}_{t+1}$ (red line) and the computed empirical intervals $\hat{I}^{b}_{t:t+1}$ (vertical grey lines). We find that $\hat{b}_{t+1}$ lies within these intervals in 207, or 91 percent, of the 228 observations.

Parameter	Value
ρ_μ	0.743
μ_+	0.047
μ_-	-0.095
ρ_b	0.738
b_+	0.805
b_-	0.262
v	0.9996
γ	0.94

Table 2: The table shows the choices of parameters.

We compute the empirical stock-price intervals $\widehat{\mathcal{I}}_t^p$ by calculating the sum in (46) for $i = 1, 2, \ldots, M$ with M chosen such that remainder terms are of order 10^{-m}, $m \geq 7$. Panel (a) in Figure 2 shows the observed stock-prices p_t (red line) and the computed empirical stock-price intervals $\widehat{\mathcal{I}}_t^p$ (vertical grey lines). We find that the stock price lies within the computed stock-price intervals in 220, or 96 percent, of the 228 observations.

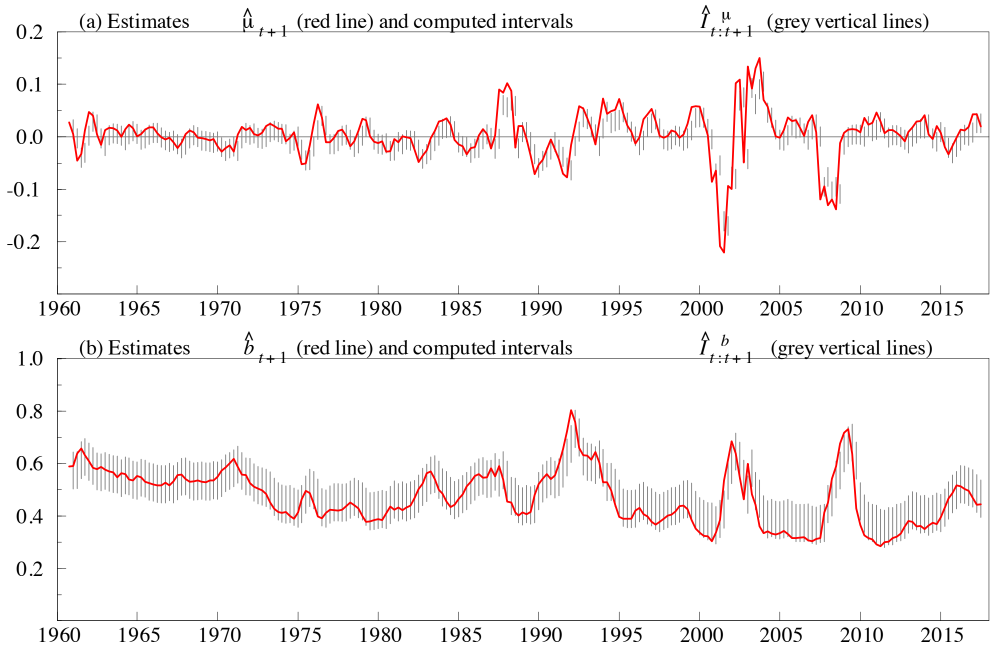

Figure 1: The figure shows the estimated time-varying coefficients and their computed intervals. Panel (a) shows the estimates $\hat{\mu}_{t+1}$ (red line) and the computed empirical intervals $\hat{I}^{\mu}_{t:t+1}$ (grey vertical lines). Panel (b) shows the estimates $\hat{b}_{t+1}$ (red line) and the computed empirical intervals $\hat{I}^{b}_{t:t+1}$ (grey vertical lines).

B.4 The Influence of Market Sentiment on Stock Prices

For sentiment data s_t described in Appendix C for the sample of 124 quaterly observations from 1984(1) to 2014(4), we compute the sentiment dependent stock price intervals in (66) and (67),

$$\mathcal{I}^p_t(s_t = 1) = \gamma \left(I^{\varphi}_{t:t+1}(s_t = 1) + I^{\phi}_t(s_t = 1) \right) x_t,$$
$$\mathcal{I}^p_t(s_t = -1) = \gamma \left(I^{\varphi}_{t:t+1}(s_t = -1) + I^{\phi}_t(s_t = -1) \right) x_t.$$

As described in Hypothesis 1 the change in the intervals depend on η, and we display here the empirical intervals for $\eta = 0.2$ and $\eta = 0.5$.

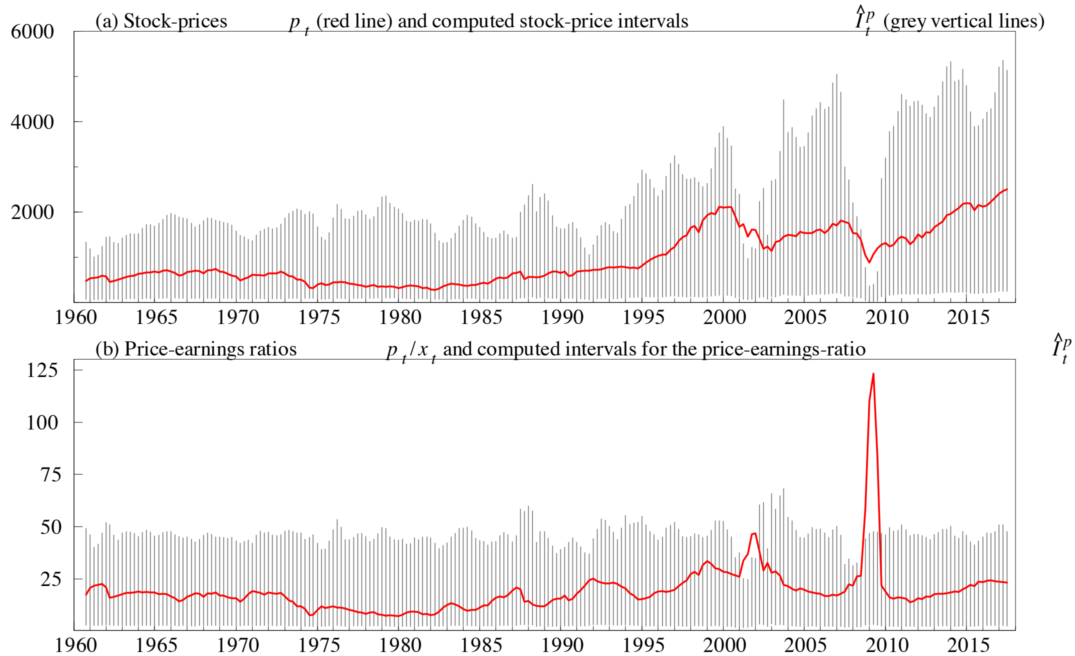

Figure 2: The figure shows the observed stock price and price-earnings ratio with their computed intervals. Panel (a) shows the observed stock price p_t (red line) and the computed empirical intervals $\widehat{\mathcal{I}}_t^p$ (grey vertical lines). Panel (b) shows the observed price-earnings ratio p_t/x_t and the computed empirical intervals $\widehat{\mathcal{I}}_t^p/x_t$ (grey vertical lines).

For $\eta = 0.2$, we find that the observed stock prices p_t lie within the computed empirical intervals $\widehat{\mathcal{I}}_t^p(s_t = 1)$ in 38 of the 39 observations where $s_t = 1$. For $\eta = 0.5$, the number reduces to 5 of the 39 observations. We find that the observed stock prices lie within the computed empirical intervals $\widehat{\mathcal{I}}_t^p(s_t = -1)$ in 36 of the 37 observations where $s_t = -1$ for $\eta = 0.2$, and in 26 of the 37 observations when $\eta = 0.5$. Figure 3 illustrates this graphically.

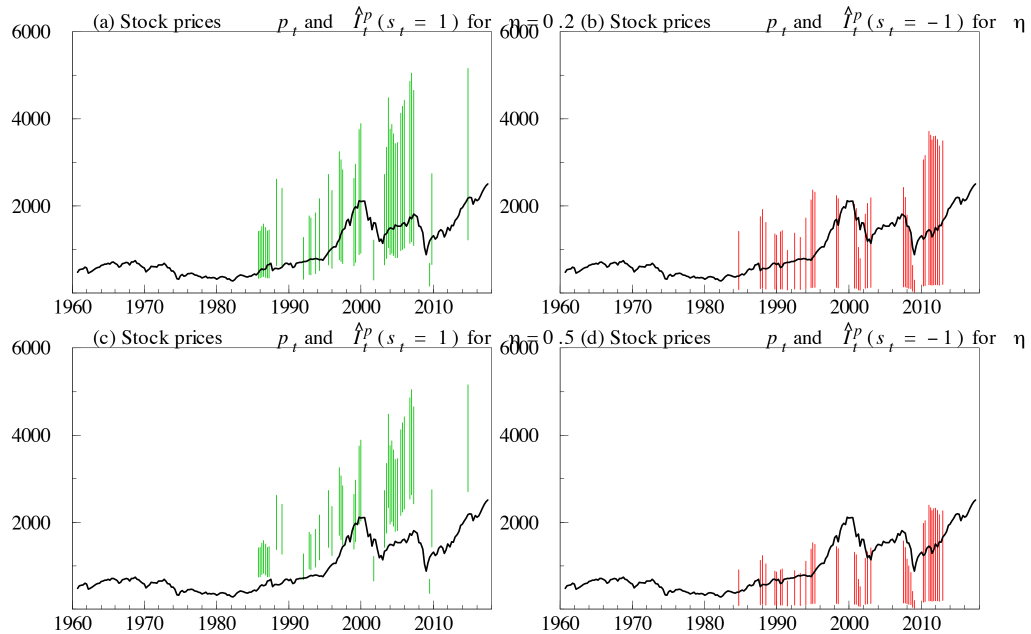

Figure 3: The figure shows the observed stock prices p_t (black lines) and the computed empirical intervals $\widehat{\mathcal{I}}_t^p(s_t)$ for $\eta = 0.2$ and $\eta = 0.5$. Panels (a) and (c) show the computed empirical intervals $\widehat{\mathcal{I}}_t^p(s_t = 1)$ (vertical green lines) for $\eta = 0.2$ and $\eta = 0.5$, respectively, for those observations where $s_t = 1$. Panels (b) and (d) show the computed empirical intervals $\widehat{\mathcal{I}}_t^p(s_t = -1)$ (vertical red lines) for $\eta = 0.2$ and $\eta = 0.5$, respectively, for those observations where $s_t = -1$.

B.5 Data Description

The data for the empirical analysis has been downloaded from Robert Shiller's website[40] in March, 2018. Real measures of the stock price index, earnings, and dividends are computed using the consumer price index (CPI). Monthly data is available, but as the earnings and dividends series are interpolated from quarterly observations we consider only the quarterly observations corresponding to March, June, September, and December.

The time-series data for real stock prices, dividends, and earnings are shown in Panels (a) and (b) in Figure 4. The log-change in real earnings is shown in Panel

[40]http://www.econ.yale.edu/~Shiller/data.htm

(c).

B.6 Definition of the Dummy Variables and Subsample Dummy Variables

The vector of dummy variables $F_{x,t}$ is defined as $F_{x,t} = \left(F^1_{x,t}, F^2_{x,t}, \ldots, F^6_{x,t}\right)'$ with:

$$F^1_{x,t} = 1\,(t = 2008\,(4))\,,\ F^2_{x,t} = 1\,(t = 2009\,(1))\,,\ F^3_{x,t} = 1\,(t = 2009\,(2))\,,$$
$$F^4_{x,t} = 1\,(t = 2009\,(3))\,, F^5_{x,t} = 1\,(t = 2009\,(4))\,,\ F^6_{x,t} = 1\,(2010\,(1) \leq t \leq 2010\,(2))\,,$$

where $1\,(\cdot)$ is an indicator variable that takes the value 1 when the expression in $(\cdot)$ is true, and zero otherwise.

Before estimating the full model in (B.1)-(B.3), the variables in $S_{x,t}$ have been selected using step-indicator saturation (SIS) with Autometrics, see Castle et al. (2015). The selection is done in the restricted model with $\omega_\mu = \alpha_{\mu 1} = \alpha_{\mu 2} = \beta_\mu = 0$ and with a target size of 0.001 in the Autometrics algorithm. The twelve subsample dummies in $S_{x,t}$ selected by Autometrics are given by the expression, $S^i_{x,t} = 1\,(\tau_{x,i} \leq t \leq \tau_{x,i+1} - 1)$ for $i = 1, 2, \ldots, 12$, where the breakpoints $\tau_{x,i}$ occur at observations $1987\,(3)$, $1988\,(3)$, $1992\,(1)$, $2000\,(4)$, $2001\,(2)$, $2001\,(4)$, $2002\,(2)$, $2002\,(4)$, $2003\,(1)$, $2004\,(1)$, $2007\,(3)$, and $2010\,(3)$, and where $\tau_{x,13} = 2017\,(4)$.

The vector of dummy variables $F_{d,t}$ is defined as $F_{d,t} = \left(F^1_{d,t}, F^2_{d,t}, F^3_{d,t}, F^4_{d,t}\right)'$ with $F^1_{d,t} = 1\,(t = 2008\,(4))$, $F^2_{d,t} = 1\,(t = 2009\,(1))$, $F^3_{d,t} = 1\,(t = 2009\,(2))$, and $F^4_{d,t} = 1\,(t = 2009\,(3))$.

Before estimating the full model in (B.4)-(B.6), the variables in $S_{d,t}$ have been selected using multiplicative-indicator saturation (MIS) with Autometrics, see Kitov and Tabor (2018). The selection is done in the restricted model with $\omega_b = \alpha_{b1} = \alpha_{b2} = \beta_b = 0$ and with a target size of $\alpha = 0.001$ in the Autometrics algorithm. The thirteen subsample dummies in $S_{d,t}$ selected by Autometrics are defined by the expression, $S^i_{d,t} = 1\,(\tau_{d,i} \leq t \leq \tau_{d,i+1} - 1)$ for $i = 1, 2, \ldots, 13$, where the breakpoints $\tau_{d,i}$ occur at observations $1972\,(3)$, $1981\,(3)$, $1991\,(1)$, $1994\,(2)$, $1999\,(4)$, $2001\,(3)$, $2003\,(2)$, $2003\,(4)$, $2007\,(4)$, $2008\,(2)$, $2009\,(4)$, $2013\,(2)$, and $2015\,(2)$, and where $\tau_{14} = 2017\,(4)$.

C The Proxy for the Market Sentiment

Based on Mangee (2017) we define the proxy for the market sentiment as follows: Define the ratio, $r_t = \frac{pos_t - neg_t}{pos_t + neg_t + 1}$, where pos_t is the number of positive words and neg_t is the number of negative words in the Wall Street Journal's "Abreast of the Market" columns for the 124 quarterly observations for the period from 1984(1) to 2014(4). The market sentiment s_t is next defined as representing optimism ($s_t = 1$) when $r_t > \tau_+$, pessimism ($s_t = -1$) when $r_t < \tau_-$, and neutral otherwise. We set the threshold values τ_- and τ_+ to the 33.3 and 66.7 percentiles of r_t. The measure r_t and the threshold values are shown in Panel (d) of Figure 4.

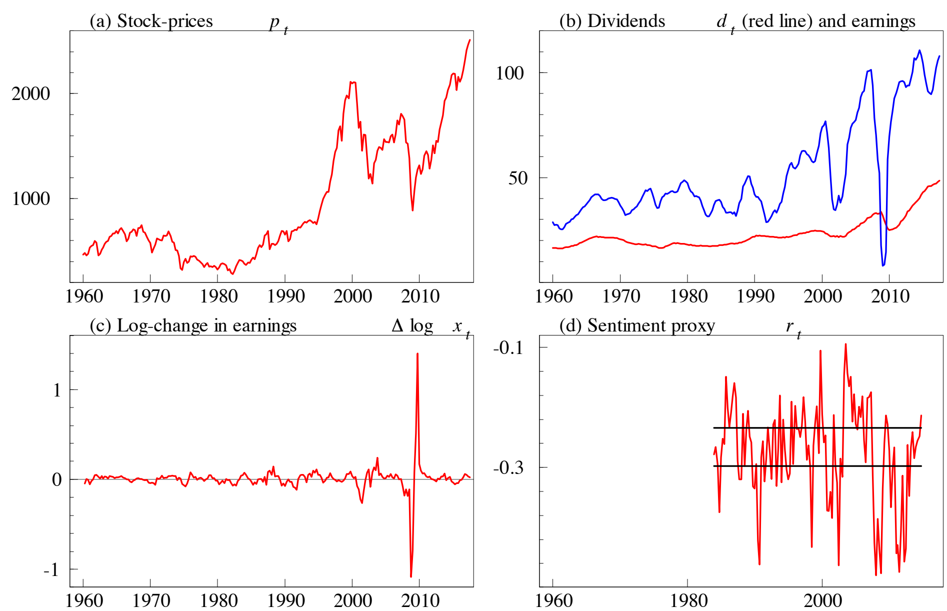

Figure 4: The figure shows the time-series data used in the empirical analysis. Panel (a) shows the stock-prices p_t, while Panel (b) shows the dividends d_t (red line) and earnings x_t (blue line). Panel (c) shows the log-change in earnings $\Delta \log x_t$. Panel (d) shows the measure r_t (red line) and the threshold values τ_- and τ_+ (black lines) used to compute the proxy for the market sentiment s_t.