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Bubbles as violations of efficient time-scales

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Abstract

It is commonly overlooked that the concept of market efficiency embowers a time-dimension. Illustrating with an example from the class of persistent random walks, we show that a price process can be a martingale on one time-scale but inefficient on another. This means that just as market efficiency can only be defined relative to an information set, it also depends on a time-scale. We use this hitherto neglected aspect to propose a new definition of bubbles that does not rely on "fundamental value": A bubble is a violation of the efficient time-scale in that the market starts to "need longer" to reflect the original information set. That is, just as excess volatility is a violation of market efficiency with respect to its filtration, bubbles are a violation of market efficiency with respect to its time-scale.

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1 Introduction

Despite an etymology that reaches all the way to the 17th century, bubbles to this day largely remain "a fuzzy word, filled with import but lacking a solid operational definition" (Garber, 2000).

Few of the definitions proffered in the literature are rigorous. As section 2, which presents a sample from the "zoo" of bubble definitions, demonstrates, many are rather vague and implicit. This often makes the definitions difficult to disentangle from the theories which they support. For example, it is not obvious if the explosiveness of expectations in rational expectations bubbles (REBs) belongs to the *explanandum* or the *explanans*. That is, are explosive expectations taken to be constitutive of bubbles or are they part of the model used to explain them? Furthermore, definitions that do not "match" each other, or are even outright inconsistent with each other, beg the question of how the results or policy implications of different studies relate to each other. Vague or incomplete definitions can make it especially difficult to see if different studies are really talking about the same thing, even if they outwardly all deal with "bubbles".

The casual manner in which bubbles are often defined has implications beyond being nettlesome. It also prevents us from solving the substantive problem of "bubble birth". Most of the literature only treats the existence or survival of bubbles under competitive market forces. That is, it asks why bubbles, once born, are not arbitraged away. But remarkably few, if any, address the question of how they originate in the first place. For example, Diba and Grossman (1987) have shown that under rational expectations, bubbles cannot arise after the start of trading. REBs, if they are to exist, must have been present since the initial trading period, or "since forever". This implies that either all markets are in a bubble (all the time) or none are (and never will).

Without a plausible mechanism of bubble birth, the important possibility that some markets are efficient while others have bubbles, or that a market may fluctuate between efficiency and bubbles, is precluded. Bubbles and efficient markets appear as constitutionally separate concepts, or competing "market paradigms". The disconnect can be partly explained by the history of economic ideas. The efficient market hypothesis essentially formalizes Hayek (1945)'s view of the market mechanism. Most bubble theories, in turn, can be viewed as variations on Minsky's financial instability hypothesis, as Brunnermeier and Oehmke (2013) have pointed out. But Minsky saw his theory as an extension of Keynes's (see, for example, Minsky, 2008). Both bubbles and efficient markets have thus been conceived in the long-running debate about the fundamental *nature* of markets (Wapshott, 2011): Markets are either self-correcting and informationally efficient or they are wild and prone to dislocations. In other words, a market either *is* a bubble (as opposed to being *in* a bubble) or it is efficient.

The question of why or how a market may *pass* from efficiency to bubbles cannot be answered until we relate bubbles and efficiency explicitly to one another. And this is not a matter of theory so much as it is one of conceptual clarification. In order to understand (or at least think about) the problem of bubble birth, we must take a step back and (re-)define both bubbles and efficiency *together*. The goal is to reframe the two as dynamic market *regimes* rather than fixed market *attributes*.

In section 3, we therefore take another look at the well-worn concept of efficient markets, with an eye on bubbles. Our main finding is slap-on-the-forehead obvious in hindsight, yet we feel justified in reporting it because it remains commonly overlooked: namely, that the concept of market efficiency embowers a time-dimension. Just as efficiency can only be defined relative to an information set, it also requires a time-scale. It is not enough to say a market "is efficient". One must also specify the time-scale at which efficiency is deemed to hold. For example, a market may be efficient on a daily scale but exhibit hysteresis on the scale of minutes, which could be exploited for arbitrage. Formally, the martingale property on one time-scale does not automatically translate down to smaller scales. It is possible for processes to be persistent on one time-scale and yet to satisfy the martingale property on a larger one. In fact, as Roll (1984) has shown, when liquidity is not infinite, the price process *has* to have some serial correlation. We think of the smallest time-scale at which the martingale property holds as informing on a market's "information processing" speed, which is inversely proportional to it, i.e. on the speed at which it absorbs and prices news correctly.

This hitherto hidden time-dimension of market efficiency provides the connecting tissue between efficient and bubble markets: Given the normal speed of a market, we (re-)define bubbles in section 4 in terms of a lengthening of the time-scale at which efficiency holds. That is, bubbles are regimes in which it takes a market significantly longer than normal to reflect the relevant information set. Formally, a bubble thus appears as a violation of the martingale property *at the original time-scale*, say t. During a bubble, the price process changes such that the martingale property now only holds on a longer time-scale, say $T > t$. To emphasize this point—that the concept of a bubble must be tied to the same time-scale t of a market as its efficiency—we also speak of a t-bubble.

Our re-definition of bubbles is sufficiently weak so as to be compatible with most of the existing definitions in the literature. The lengthening of the time-scale only serves to create space for a variety of bubble dynamics "in-between" the points at which the martingale property is restored. At the most basic level, the *explanandum* of a bubble, as defined by this shift in scales, can be modeled by a change in memory of the price process. The task of a bubble *explanans* then would be to explain how this change in memory comes about. This is the approach we have taken in Sohn and Sornette (2016). Regardless of the dynamics at play, the mere act of locating efficient markets and bubbles at opposite ends but on a common scale now makes transition dynamics possible and thus brings the problem of bubble birth into sharper focus than before. We view this as the main contribution of this paper.

2 Bubble definitions

Fama once famously vented that "I didn't renew my subscription to The Economist because they use the word bubble three times on every page. Any time prices went up and down – I guess that is what they call a bubble."[3] Polemic as it is, the complaint is not entirely without foundation. Even a quick dive into the bubble literature invariably ends in a bog of definitions. In the following, we present a small sampling. To aid our discussion, we have bunched them into three broad categories.

Statistical definitions As a first group of bubble definitions, there are those that focus on the price trajectory or other observables such as trading volume, without reference to theoretical notions like fundamental value. For example, Kindleberger and Aliber (2005) regard as bubbles "any upward price movement over an extended period of fifteen to forty months that then implodes." Exchanging the specification of the time-horizon for a size requirement, as it were, Goetzmann (2017) defines bubbles as a doubling in the market price followed by a 50% fall[4]. Presumably, then, bubbles cannot occur in fixed income or other markets where there is a natural upper bound on the market price! The fund manager GMO proposes that bubbles occur "when prices rise two standard deviations above their norm."[5] This is more flexible than an absolute size requirement but, alas, opens a whole other can of worms, like estimation issues, ergodicity assumptions, or the question of whether the second moment even exists for a given asset.

Brock, as cited in Veres (2013), defines bubbles as "a monotonically increasing sequence of prices." Hüsler et al. (2013) and Leiss et al. (2015) cite super-exponential growth rates[6] as the hallmark of a bubble. This chimes

[3]Quoted in Cassidy (2010).

[4]One may recall here that Black (1986) defined efficient markets as ones "in which price is within a factor of 2 of value. [. . .] The factor of 2 is arbitrary, of course."

[5]Mackintosh (2014), Buttonwood (2017)

[6]faster than exponential growth, or growth rates that themselves grow

with Kindleberger and Aliber (2005) in that it also implies an unsustainable price path but differs in that it does not require an "implosion" or market crash.

What the definitions in this category have in common is that they neither imply nor necessitate a mispricing *per se*. They focus on the observable (the price series) and do not mix theoretical concepts into the definition. In particular, there is no notion of value here. This is an appealing feature for a definition, as *explanandum* and *explanans* then are clearly separated from each other. Bubbles, defined like this, can be tested without the problem of the joint hypothesis. On the downside, insofar as a definition depends on the full path, including a crash at the end, it can be guilty of *post hoc ergo propter hoc* in practice. Insofar as the theoretical underpinning is lacking, the definitions in this category can also be too broad in scope: Empirically, too many price series can fit a statistical bubble definition without necessarily corresponding to our intuition of what a bubble "should" be. For example, an interest rate sensitive stock might follow a rate cycle up "over an extended period of fifteen to forty months" only to then "implode" upon the revelation of a criminal investigation. Few would characterize this as a bubble. Context, as it were, *is* important.

Comparative definitions As a second category, there are bubble definitions based on comparisons, usually between price and some notion of value. For instance, the *New Palgrave Dictionary of Economics* defines bubbles as "asset prices that exceed an asset's fundamental value" (Brunnermeier, 2008). Bland as it may appear, this excludes the possibility of negative bubbles, a significant restriction to make *by definition*, as it were. Temin and Voth (2004) by contrast identify bubbles as "periods of substantial mispricing" which allows for undervaluations as well as overvaluations but adds a size requirement ("substantial"). Levine et al. (2014) define bubbles as simply a "misfit between the market price and the true value of an asset" with no such qualification. This lack of specificity makes it hard to see where the line between excess volatility and bubbles should be drawn. The point is not to niggle or read too much into what may have been intended as merely passing remarks in a much longer work. It is to show that just because a definition is done casually does not mean it has no consequences—especially when we have to relate different studies to each other.

Apart from direct appeals to value, comparisons can also refer, more obliquely, to the information sets on which "true value" is presumably based. For instance, Blanchard and Watson (1982) define bubbles as price movements which are "unjustified by information available at the time." More emphatically, Asness (2014) demands that the term should apply only when "no reasonable future outcome can justify" the price. This seems to posit a range of admissible price paths, defining bubbles negatively, or by exclusion.

For all their differences, comparative definitions always require a theory of asset pricing, if only implicitly, for a notion of what the correct price is supposed to be. This is their Achilles' heel and the chief criticism of efficient market proponents. For example, Santos and Woodford (1997) compare the market price of an asset to the state-price weighted sum of its real payoffs, while Siegel (2003) uses the realized return on an asset over a sufficiently long time after trading. Different studies can thus agree, in general terms, to define bubbles as a divergence of price from value *and still disagree* over whether a particular price series is a bubble or not. It all seems a bit arbitrary, confirming the suspicions of those who think that the very notion of bubbles is jerry-rigged.

Detailed definitions A third group of definitions goes beyond the perceived gap between price and value by tying it to specific explanations. For example, Kirman and Teyssière (2002) require that the gap between price and value be "endogenous, i.e., not directly produced by exogenous shocks." In other words, the mispricing must arise in a certain way in order for it to count as a bubble. Brunnermeier and Oehmke (2013) concur that "not every temporary mispricing can be called a bubble." In particular, it has to arise "because investors believe they can sell the asset at an even higher price to some other investor in the future," so for them the speculative motive is essential. Roubini (2006) even introduces a policy dimension by distinguishing between "endogenous" and "exogenous" bubbles, where the former are bubbles whose "probability and size can be affected by monetary policy" while the latter cannot[7]. As an extreme example of the involute nature of the definitions in this category, let us quote from Shiller (2014):

> I would say that a speculative bubble is a peculiar kind of fad or social epidemic that is regularly seen in speculative markets; not a wild orgy of delusions but a natural consequence of the principles of social psychology coupled with imperfect news media and information channels. [...] I offered a definition of bubble that I think represents the term's best use: A situation in which news of price increases spurs investor enthusiasm which spreads by psychological contagion from person to person, in the process amplifying stories that might justify the price increases and bringing in a larger and larger class of investors, who, despite doubts about the real value of an investment, are drawn to it partly through envy of others' successes and partly through a gambler's excitement.

Basically the obverse to our first category, it is not surprising then to find that detailed definitions tend to be too narrow in scope. Would a bubble that arose by a different mechanism, or in a market in which the proposed mechanism does not apply, also be a "bubble"? For example, would a "political bubble" (McCarty et al., 2013) not count as a bubble to Brunnermeier and Oehmke (2013)? Or if it did, doesn't this mean that there must exist a less restrictive superset of bubbles, of which the two variants (political vs. speculative) are but particular cases? And if not, how are we to relate the results and policy implications of different studies to each other? Would a bubble

[7] A similar but more general argument, less focused on monetary policy, has been put forth in Johansen and Sornette (2010).

indicator constructed for, say, speculative bubbles still be expected to detect politically driven ones?

The above quote also illustrates that the more detailed a definition, the more likely it is to mesh the notion of bubbles with behavioral assumptions or market frictions. Arguably it is this that makes bubbles such a loaded term. With respect to recessions, inflation or unemployment, the debates may be vigorous but at least their subjects are accepted. By contrast, bubbles remain existentially controversial. Perhaps this is because the more detailed a definition, the more it acts as a Trojan horse: the mere use of the term may already admit of assumptions one does not wish to make. It is thus that the rejection of behavioral hypotheses or doubt about the effectiveness of monetary policy may lead one to reject the concept of bubbles, almost as an unintended side effect. For the sake of discussion, we should therefore move away from such evocative definitions towards greater formalism and pithiness. In the words of Brock (2014), "for the quality of a theory to improve over time, definitions must become more rigorous and less ambiguous."

3 Efficient markets

Since Hayek (1945), the view of markets has shifted from their allocative function (as emphasized by the early classics) to a conception of markets as information processing machines[8]. In this more modern view, markets transform informational inputs, modeled by a filtration $(\mathcal{F}_t)_{t=0}^{\infty}$, into price signals $(p_t)_{t=0}^{\infty}$. Markets thus act as a map $\Delta\mathcal{F} \to \Delta p$ from news to price changes. Market attributes are naturally defined in terms of these primitives. Eliding the discount factor for simplicity, the efficiency of markets has been characterized by the martingale property (cf. Samuelson, 1965, 1973, LeRoy, 1989), where

$$\mathbb{E}_t(p_{t+1}|\mathcal{F}_t) = p_t \tag{1}$$

[8]Cf. Mirowski (2002) or Gleick (2011)

That is, the market is said to be efficient *relative to the news process* $(\mathcal{F}_t)_{t=t_0}^{\infty}$ iff the map $\Delta\mathcal{F} \to \Delta p$ produces a martingale. Economically, this means that price is an unbiased predictor and that the price history cannot be exploited for (excess) profit[9] because the increments Δp cannot be serially dependent. A related perspective is that an efficient market does not allow trading profits based on the current information set (Jensen, 1978)[10]:

> A market is efficient *with respect to information set* θ_t if it is impossible to make economic profits by trading on the basis of information set θ_t. By economic profits, we mean the risk adjusted returns net of all costs. Application of the zero profit condition to speculative markets under the assumption of zero storage costs and zero transactions costs gives us the result that asset prices (after the adjustment for required returns) will behave as a martingale with respect to the information set θ_t.

Standard as this conception of market efficiency is, it embowers an aspect that is often overlooked: the time dimension. It is implicitly understood in equation (1) that t is the relevant time-scale. That is, if we take t_0 to be the present, equation (1) can be written out like

$$\mathbb{E}_{t_0}(p_{t_0+n\times t}|\mathcal{F}_{t_0}) = p_{t_0} \tag{2}$$

with $n = 1$ and the understanding that the martingale condition holds for $n \in \mathbb{N}$. To see where we are leading with this somewhat awkward transcription, think of any given discrete-time price process as merely a sampling from an underlying continuous-time process. But time is infinitely divisible and the same price process can be sampled at different rates or frequencies, for example $\tau < t$. It is true that the finite-dimensional distributions of a process determine, in their totality, the distribution of a process[11]. However, the mere

[9]Where "excess" means returns above those expected under the stochastic discount factor or systematic risk premia, see Lucas (1978).

[10]Our emphasis

[11]As long as the consistency conditions in Kolmogorov's existence theorem are satisfied, see Billingsley (1999, thm. 13.6).

fact that a process is a martingale on one time-scale neither necessitates nor implies that it is one on another.

This opens the possibility that a market is efficient on one time-scale but inefficient on another. Such a disjunction between time-scales can be supported empirically[12] as well as theoretically, from reading Roll (1984) "in reverse": To recap his argument, as long as liquidity is not infinite and there is a strictly positive bid-ask spread $s > 0$ in the market, successive price changes Δp will exhibit serial dependence and the martingale property will not hold. Adapting his notation, let those price changes be measured at the time-scale $\tau < t$, i.e. $\Delta p_{2\tau} = p_{2\tau} - p_{\tau}$, to make the connection to our discussion clearer. The bid-ask spread induces an asymmetry in the price path at the scale τ (see figure 1): If the last transaction was conducted at the bid, then the next move can only be up (by the spread s) or 0. If the last transaction was conducted at the ask, then the next move can only be down (by s) or 0. One time-step further, the situation is reversed. If the last move was up or 0 (down or 0), then the next move can only be down or 0 (up or 0). The bid-ask spread thus introduces a serial dependence into successive price movements that is not compatible with the martingale condition of an efficient market.

[12]See for example Barany and Beccar Varela (2012) or, more plastically, the case study of Maloney and Mulherin (2003). The point being that a market, no matter how efficient, always needs *some* time to digest information.

Δp_{t+1}	p_{t-1} is at the bid: $\Delta p_t = 0$	p_{t-1} is at the bid: $\Delta p_t = +s$	p_{t-1} is at the ask: $\Delta p_t = -s$	p_{t-1} is at the ask: $\Delta p_t = 0$
$-s$	0	¼	0	¼
0	¼	¼	¼	¼
$+s$	¼	0	¼	0

Figure 1: Table of transition probabilities, conditional on the last transaction having been conducted at the bid or at the ask price, adapted from Roll (1984, p. 1129)

At the same time, over a sufficient number n of time-steps τ, the transition probabilities converge to a (symmetric) steady state. This means that for $t \geqslant n\tau$, with n sufficiently large, the effect of the bid-ask spread (or, by extension, other microstructural factors) "washes out": Measured on the micro-scale τ, the process exhibits serial dependence; measured on the macro-scale $t \geqslant n\tau$, the price process can conform to the martingale property again.

Let us illustrate this phenomenon analytically with a toy model, the two-step random walk in Arneodo and Sornette (1984), a special case of the class of persistent random walks (cf. Rudnick and Gaspari, 2004, sec. 5.2). Let $\Delta p \in \{U, D\}$ for up = +1, down = -1. Define π_{UU} as the joint probability that the price goes up twice in a row; π_{UD} as the probability that an up move is followed by a down move; and π_{DD}, π_{DU} as the probabilities of down-down and down-up moves. Let $\pi_{UU} = 1/6, \pi_{DU} = \pi_{UD} = 1/3, \pi_{DD} = 1/6$. Suppose the last move was up and start at time t_0 with $p_{t_0} = 100$. Then

$$\begin{aligned}
\mathbb{E}(p_{t_0+\tau}|\mathcal{F}_{t_0}) &= \mathbb{E}(p_{t_0+\tau}|\text{up}) && (3)\\
&= 100 + \pi_{U|U} \times 1 + \pi_{D|U} \times (-1) && (4)\\
&= 100 + \frac{1}{3} - \frac{2}{3} && (5)\\
&\neq 100 && (6)
\end{aligned}$$

where $\pi_{U|U}, \pi_{D|U}$ are the corresponding conditional probabilities. That is, one time-step forward, this two-step random walk is not a martingale. However, if we perform the same calculation two time-steps forward,

$$\begin{aligned}
\mathbb{E}(p_{t_0+2\tau}|\mathcal{F}_{t_0}) &= \mathbb{E}(p_{t_0+2\tau}|\text{up}) && (7)\\
&= 100 + \pi_{UU|U} \times 2 + \pi_{DD|U} \times (-2) && (8)\\
&= 100 + \frac{1}{3} - \frac{1}{3} && (9)\\
&= 100 && (10)
\end{aligned}$$

The reason is that the memory gets lost at the time-scale $t = 2\tau$,

$$\begin{aligned}
\pi_{UU|U} &= \frac{\pi_{UU\wedge U}}{\pi_U} && (11)\\
&= \frac{\pi_{UU} \times \pi_U}{\pi_U} && (12)\\
&= \pi_{UU} && (13)
\end{aligned}$$

As a result, even though the same price process exhibits serial correlation at the scale τ, it conforms to the martingale property at the scale $t = 2\tau$.

To sum up, market efficiency has a time-dimension. It is therefore not enough to speak of a market as efficient. In addition to the news process $(\mathcal{F}_t)_{t=0}^{\infty}$ relative to which efficiency is defined, one also needs to state *at which time-scale* efficiency is supposed to hold. The time it takes a market to fully absorb an information increment $\Delta\mathcal{F}$ can be random but has a characteristic scale, in the sense that it fluctuates within certain bounds or that its mean is defined. In the following, we will take this characteristic time-scale of a market as a given[13] and call it t.

[13] It is also possible, though, to conceive of financial markets in which the mean time to digest news diverges. This could occur, for instance, when the absorption time is distributed according to a power law in the tail with tail exponent less than 1. As many response functions are power laws in the time domain with small exponent, this is indeed an interesting possibility. In this case, the market would never be efficient even at arbitrarily large time-scales.

4 Bubbles

In light of the above, our new definition of bubbles needs no further ado. In order to emphasize that the concept of a bubble must be tied to the same time-scale t of a market as its efficiency, we speak of t-bubbles.

Definition 4.1. *Given a market that is efficient relative to $\mathcal{F}$ at the time-scale t, a t-bubble occurs when the price process changes such that the martingale condition $\mathbb{E}_{t_0}(p_{t_0+T}|\mathcal{F}_{t_0}) = p_{t_0}$ now only holds at time-scales $T > t$. As a boundary case, we include regimes where $T = \infty$ or the condition never holds.*

Colloquially, we may call t-bubbles simply "bubbles" so long as it is clear that the notion of a bubble only makes sense when set in relation to the "normal speed" $\sim 1/t$ of the market in which it is to occur. A bubble is a slowdown in the map $\Delta\mathcal{F} \to \Delta p$, a sort of "informational constipation" of the market if you will, and a slowdown needs a reference point. Just as market efficiency cannot be defined in a vacuum but only relative to a news process $\mathcal{F}$ at a time-scale t, the bubble definition we propose depends on the benchmark of an efficient market.

Moving the focus to time-scales allows us to purge all reference to "fundamental value" from the *explanandum*, just as there are no more behavioral traders or speculative motives or market frictions left in it, all of which we believe to belong more properly to the *explanantes* proposed. Yet for all this, or should it be because of this, our re-conception of bubbles is weak enough to be compatible with the literature. The general principle is to eliminate (the conditions for) the bubble from a model and inspect the time-scale t at which the market in the model is efficient. If the bubble component has a mean survival time, this is the lower bound for T. For example, under the limited arbitrage argument of Abreu and Brunnermeier (2003), the duration of the bubble is finite with a survival time of $\bar{\tau}$ (in their notation). Without the bubble, the market is efficient at the time-scale t; with a bubble, it slows down with a longer required time-scale $T = \bar{\tau}$.

The class of rational expectations bubbles constitutes an interesting example because even under a bubble the price path of, for instance, Blanchard (1979) still follows a martingale. In equilibrium, the probability of a crash is supposed to exactly balance the added growth factor of the bubble component $b_t = p_t - \bar{p}$, where $\bar{p}$ is the fundamental value, or

$$\mathbb{E}_t(b_{t+1}|b_t > 0) = b_t \tag{14}$$

if we elide the discount factor for simplicity. That is, the price would simply incorporate the bubble component via

$$\mathbb{E}(p_{t+1}|\mathcal{F}_t) = \bar{p} + b_t \tag{15}$$

How are we to make sense of definition 4.1 in light of this? We said above that the efficiency of a market depends, conceptually, on the specification of both an information process and a time-scale. Definition 4.1 in turn relies on an efficient market as a benchmark. To fit the important class of rational expectations bubbles with our new definition, note that the information set (or filtration) $\mathcal{F} = (\mathcal{F}_t)_{t=0}^{\infty}$ contains, or is generated by, all the fundamental variables *as well as* the bubble component $b = (b_t)_{t=0}^{\infty}$. Therefore, the bubble according to 4.1 cannot be defined relative to $\mathcal{F}$. Instead, we must introduce a "copy" of the market, a hypothetical in which all the elements are the same (agents and their preferences, assets, institutions, etc.) *except* the information process, which should not contain, or be generated by, the bubble component b_t. Let us call this filtration $\mathcal{G} = (\mathcal{G}_t)_{t=0}^{\infty}$. It is only against this hypothetical efficient market against which the bubble in Blanchard (1979) can properly be defined in accordance with 4.1.

$$\mathbb{E}(p_{t+1}|\mathcal{G}_t) = \bar{p} \tag{16}$$

$$\neq \bar{p} + b_t \tag{17}$$

That is, relative to the efficient market, the bubble component introduces an estimation or valuation error which survives with probability π and collapses with probability $1-\pi$. It thus has an expected length of $\pi/(1-\pi)$ time-steps of scale t, or $T \geqslant \pi/(1-\pi) \times t$. For example, if $\pi = .95$, then $T \geqslant 19t$. As the probability $(1-\pi)$ of a crash approaches zero, $T \to \infty$ in the limit.

The example of rational expectations bubbles indicates a link between time-scales and information processes. Taking the efficient market as a baseline, if we change the information process from $\mathcal{G}$ (containing only fundamental variables) to $\mathcal{F}$ (including knowledge of the bubble), the price continues to follow a martingale at the original time-scale t. If we leave the information process unchanged from the benchmark of an efficient market, then the time-scale at which the martingale property is restored expands to $T > t$.

This link between the information process and the characteristic time-scale of a market leads to an alternative representation of our bubble definition that leaves the time-scale untouched. We mention it here only in passing as it needs to be more fully developed in a separate paper. Even an idealized efficient market errs by $\xi_{t+1} = p_{t+1} - \mathbb{E}_t(p_{t+1}|\mathcal{F}_t)$. These errors are by assumption (or under the efficient market hypothesis) independent, and we can further assume identically distributed. As a consequence of this iid assumption, any subsample average of the errors will be close to its mean 0. One can quantify this adherence of the subsample averages, or violations thereof, with a statistic called *long strange segments (LSS)*, first studied by Erdös and Rényi (1970) and more recently by Samorodnitsky and his collaborators (Samorodnitsky, 2006, Mansfield et al., 2001, Rachev and Samorodnitsky, 2001).

Definition 4.2. *Given an ergodic, stationary stochastic process* $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots)$ *with finite mean* $\mu = \mathbb{E}\xi_1$ *and* $\theta \in \mathcal{B}$*, the Borel-algebra over* $\mathbb{R}$*, we define for every* $n = 1, 2, \ldots$

$$R_n(\theta, \boldsymbol{\xi}) = \sup \left\{ t_j - t_i : 0 \leqslant t_i < t_j \leqslant n, \frac{\xi_{t_i+1} + \cdots + \xi_{t_j}}{t_j - t_i} \in \theta \right\} \tag{18}$$

with $R_n = 0$ *if the supremum is taken over the empty set. Given a* θ *such that* $\mu \notin \theta$*, we call the subsequence* $(\xi_t)_{t=t_i}^{t_j}$ *a* long strange segment (LSS).

That is, long strange segments indicate the longest subperiods in which the law of large numbers seems to be temporarily suspended. The farther away

θ is from $\mu = 0$ (in our case) or the longer $t_j - t_i$, the "stranger" the segment or its occurrence in the sequence. The functional thus captures exactly the usual intuition about bubbles, namely the *deviation* from a natural reference value over a *sustained* period of time.

It is worth emphasizing here that the deviations are *not* relative to "fundamental value" but to the benchmark of the efficient market: The sequence of $(\xi_t)_{t=0}^{\infty}$ stem from the presumption of a martingale price process. This in turn means that the presumed statistical properties of $(\xi_t)_{t=0}^{\infty}$ are specific to the filtration $(\mathcal{F}_t)_{t=0}^{\infty}$ and the time-scale t that were used to pin down the efficiency of the market. Usually it is assumed that only variables that affect the fundamentals or the dividend process are part of the information process $(\mathcal{F}_t)_{t=0}^{\infty}$. Suppose now that the actual price process contained a "bubble component" as in Blanchard (1979), i.e. a variable that is orthogonal to the dividend process, say $(b_t)_{t=0}^{\infty}$. Let us write $(\mathcal{G}_t)_{t=0}^{\infty}$ for an information process that contains this bubble component and define the concomitant error sequence by $\eta_{t+1} = p_{t+1} - \mathbb{E}(p_{t+1}|\mathcal{G}_t)$. Now it is $(\eta_t)_{t=0}^{\infty}$ that is iid, with correspondingly small and bounded LSS.

By contrast, since the bubble component is outside $(\mathcal{F}_t)_{t=0}^{\infty}$ relative to which market efficiency was originally defined, it enters the error terms $(\xi_t)_{t=0}^{\infty}$ and thus bounds them away from $\mu = 0$: $\xi_t = \eta_t + b_t$. Setting $\theta = (0, \infty)$, $R_n(\theta, \boldsymbol{\xi})$ scales with n and becomes orders of magnitude larger than under the efficient market regime (i.e. without the bubble component). This difference between the behaviors of ξ_t and η_t motivates our alternative representation of bubbles in terms of errors:

Definition 4.3. *A* bubble *is a market regime (=a set of conditions) during which the long strange segments $R_n(\theta, \boldsymbol{\xi})$ of the error terms $\xi_t = p_t - \mathbb{E}_{t-1}(p_t|\mathcal{F}_{t-1})$ with $0 \notin \theta$ grow faster (as a function of n) than under conditions of market efficiency.*

5 Conclusion

During our review of the bubble literature (Sohn and Sornette, 2017), we noticed that most of it only treats the existence or survival of bubbles under competitive market forces but cannot explain how they arise in the first place. This omission is curious for its prevalence, given the import of the question. We think it stems from the insufficiency of previous bubble definitions, in particular from the lack of connection between the concepts of bubbles and efficient markets. In a sense, the difference between the two was too jarring. Efficient markets and bubbles seemed like separate "market paradigms", rather than opposite ends on a common scale.

In this article, we have argued that it is time, or memory, that provides the connecting tissue between the two concepts. The martingale condition expresses the idea that the information relative to which efficiency is defined cannot be exploited for easy profit. Obversely, it tells us that new information *will* find its way into prices. The time-scale at which the condition holds tells us how fast. If something interferes with the market such that new information is not reflected in prices, or takes much longer than before until it is, we speak of a bubble. This new perspective makes it much more natural to think of reasons why or how a market may transition from one state to the other. We view this as one of the main advantages of our definition.

Another problem our new definition addresses is the limitation of many bubble models to partial equilibria. Take, for example, the explosiveness of expectations in rational expectations bubbles. Not only does this seem to exclude the entire credit market, which is much larger and more important than the stock market. As Kurz (2015) argues, it also does not translate well into a general equilibrium setting because the bubble cannot stay isolated. It must eventually capture all assets and prices in the economy, at which point the concept of a bubble loses its force if not its entire meaning. A similar argument has been made by Loewenstein and Willard (2006) for (or rather against) noise trader bubbles where very mild assumptions about the availability of a riskless asset diminish the force of the noise traders and thus the possibility of bubbles as defined therein.

Our definition not only allows for an easier birth but also an easier life of bubbles under competitive market forces. The reason is that a slowing-down of the information processing capacity of a market is a relatively mild condition. If all markets need time to transform news into prices, then some markets can take longer than others without acting as a wrecking ball to the entire price system (as bubbles with explosive price expectations do).

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